PROPERTIES OF ANTI DE SITTER BLACK HOLES

A Project Report Submitted

in Partial Fulfilment of the Requirements

for the Degree of

MASTER OF SCIENCE

in

Physics

by

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to

INDIAN INSTITUTE OF TECHNOLOGY

INDORE - 452 020, INDIA

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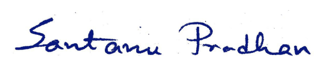
DECLARATION

I, **Santanu Pradhan (Roll No: 2203151019)**, hereby declare that, this report entitled **"Properties of Anti de-Sitter Black Holes"** submitted to Indian Institute of Technology Indore towards the partial requirement of **Master of Science** in **Physics**, is an original work carried out by me under the supervision of **Dr. Manavendra Mahato** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold academic ethics and honesty. Whenever a piece of external information or statement or result is used then, that has been duly acknowledged and cited.



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CERTIFICATE

This is to certify that the work contained in this project report entitled **"Properties of Anti De Sitter Black Holes"** submitted by **Santanu Pradhan** (**Roll No: 2203151019**) to Indian Institute of Technology, Indore towards the partial requirement of **Master of Science** in **Physics** has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree.

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December 2023

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ABSTRACT

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The AdS/CFT correspondence, a groundbreaking theoretical framework in modern physics, has provided profound insights into the unification of quantum field theory and gravity. This project delves into the AdS/CFT correspondence and focuses on its application to the intriguing realm of anti-de Sitter (AdS) black holes. Anti-de Sitter space, characterized by its negative cosmological constant, serves as a valuable backdrop for studying the interplay between quantum field theory and gravity in a highly curved spacetime.

This research project aims to elucidate various properties of AdS black holes and their connections to conformal field theories (CFTs) in lower dimensions. We investigate the thermodynamic, holographic, and information-theoretic aspects of these black holes, aiming to unravel the mysteries that lie at the intersection of quantum mechanics and gravity.

This project is divided into two parts - first, we will study AdS/CFT correpondence and perform some calculations to prepare our tools and second we will study properties of anti de sitter black holes and why we do bother about them.

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Chapter 1

Introduction

In the realm of theoretical physics, there exist profound and captivating ideas that transcend the boundaries of our conventional understanding of the universe. These ideas not only challenge our fundamental notions of space, time, and matter but also reveal unexpected connections between disparate facets of the physical world. Among these remarkable concepts are the AdS/CFT correspondence, holographic duality, and large N symmetries. Each of these notions, in its own right, has revolutionized our comprehension of the cosmos. Yet, when viewed together, they present a panoramic view of the universe that defies our intuitions and beckons us to explore the hidden depths of theoretical physics.

The AdS/CFT correspondence, often referred to as the holographic principle, stands as a pinnacle of 21st-century theoretical physics. It asserts a remarkable relationship between two seemingly unrelated theories: Anti-de Sitter space (AdS), a negatively curved spacetime, and conformal field theory (CFT), a quantum field theory living on its boundary. In this duality, a gravitational theory in AdS is equivalent to a quantum field theory on the boundary of that spacetime. This profound insight, first conjectured by Juan Maldacena in 1997, not only bridges the chasm between quantum mechanics and gravity but also opens new vistas for understanding black holes, quantum entanglement, and the very nature of spacetime itself.

Complementing the AdS/CFT correspondence is the concept of holographic duality. This overarching idea, inspired by the AdS/CFT correspondence, extends the notion of holography to a broader context, suggesting that our three-dimensional universe might be encoded on a lower-dimensional boundary. Holographic duality challenges our perception of reality, hinting at the possibility that the entirety of our universe's physics can be represented on its cosmic horizon.

Further enriching this tapestry of ideas is the gauge-gravity correspondence, which explores the connection between gauge theories—such as quantum chromodynamics (QCD)—and gravity. This correspondence reveals that certain strongly coupled gauge theories can be described by gravitational theories in higher dimensions. This revolutionary insight has implications not only for understanding the strong force in particle physics but also for elucidating the behavior of quark-gluon plasmas, shedding light on the early moments of our universe.

My project embarks on a captivating journey into the interconnected worlds of the AdS/CFT correspondence, holographic duality, and gauge-gravity correspondence. We will delve into the historical developments, mathematical foundations, and multifaceted applications of these ideas, striving to unravel the mysteries they hold and the profound implications they bear for our understanding of the cosmos.

In November 1997 Maldacena wrote his groundbreaking paper, conjecturing the relation which became known as the AdS/CFT correspondence. In this report we will explain how Maldacena arrived at this remarkable conjecture, indicate its formulation and key features, and mention its immediate sociological impact. Since the statement is perhaps as mystifying as it is profound, I give a more modern perspective which motivates the correspondence with a course on string theory. I then build up the basics of the AdS/CFT dictionary which simultaneously exemplify some of the initial checks of its , the latter focusing on the important context of AdS black holes. I refer to the excellent early review by 'MAGOO' .

1.1 Hints For Holography

"The carrer of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction." - **Sidney Coleman**

In this chapter,we will get a favor of the holographic duality. We first study gravity system and derive blackhole thermodynamics where holography principle emerges. Then we investigate gauge theory in the large N (t'Hooft)limit. At last, we compare such a theory and the string theory and give hints on holographic duality.

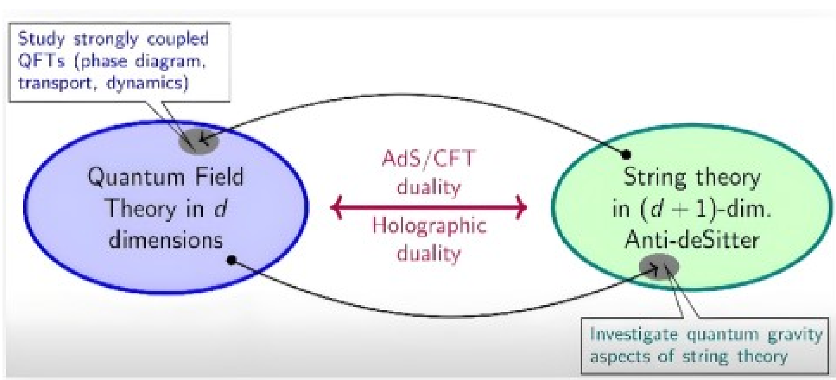
Anti de sitter space generalises to any number of space dimensions. In higher dimensions, it is best known for its role AdS/CFT correspondence, which suggests that it is possible to describe a force in quantum mechanics (like electromagnetism, the weak force or strong force) in a certain number of dimensions with a string theory where the strings exist in an anti de sitter space with one additional (non - compact) dimension.

1.1.1 Prelude

The basic theoretical structure of interactions is well understood by path integral formalism plus Wilsonian Renormalization Group. Then any calculation can be reduced to this algorithm, although it does not mean we can necessarily perform it. On the other hand, gravity is quite different. From theory of general gravity (GR):

Classical Gravity = Space-time (no concept of spacetime fabric, simply GR)

However, how do we understand quantum gravity? Spacetime here should become dynamical. There are many puzzling questions. Is spacetime fundamental or emergent? Is it continuous or discrete? What is the quantum nature of blackholes? How did the universe begin? One intriguing feature about gravity is that it is the weakest interaction, which may be a fundamental aspect. In 1997, Juan Maldacena discovered the famous duality:



1.1.2 Emergence of Gravity

From field theory perspective, it is natural to ask whether massless spin-2 particles (gravitons) can arise as bound states in a theory of massless spin-1 (photons,gluons) and spin-1/2 particles (protons,electrons). If the answer is yes, we can conclude that gravity can be emergent. For example, in Quantum Chromodyanmics(QCD),there are indeed massive spin-2 excitations. Could one tweak such a theory that mass-less spin-2 particles emerge?Such hopes were however dashed by a powerful theorem of Weinberg and Witten.

Theorem 1.1.1. *A theory that allows the construction of a Lorentz-covariant conserved 4-vector current* J^μ *cannot contain mass-less particles of spin* $> 1/2$ *with non-vanishing values of the conceived charge* $\int J^0 d^3x$

Theorem 1.1.2. *A theory that allows a conserved Lorentz-covariant stress tensor* $T^{\mu\nu}$ *cannot contain mass-less particles of spin* > 1.

Proof. Suppose we have such a theory that allows Lorentz covariant conserved current and stress tensor, and there exist massless particles of spin-J. One-particle state are denoted as $|k,\sigma\rangle$, $k^\mu = (k^0, \mathbf{k})$, $\sigma = \pm j$ (helicity). We have

$$\hat{R}(\theta, \hat{k}) |k,\sigma\rangle = e^{i\sigma\theta} |k,\sigma\rangle \tag{1.1}$$

Where $\hat{R}(\theta, \hat{k})$ is the rotational operator by an angle θ around $\hat{k} = \frac{\mathbf{k}}{|k|}$. The conserved Lorentz covariant current is J^μ, with the conserved charge

$$\hat{Q} = \int J^0 d^3x \tag{1.2}$$

and the Lorentz covariant stress tensor is $T^{\mu\nu}$, with the conserved momentum

$$\hat{P} = \int T^{0\mu} d^3x \tag{1.3}$$

Then,

$$\hat{P}^{\mu} |k, \sigma\rangle = k^{\mu} |k, \sigma\rangle \tag{1.4}$$

If $|k, \sigma\rangle$ is charged under the symmetry generated by J^{μ} with charge q:

$$\hat{Q} |k, \sigma\rangle = q |k, \sigma\rangle \tag{1.5}$$

We want to show that:

1. if $q \neq 0$, $j \leq \frac{1}{2}$
2. $j \leq 1$, for Lorentz covariant conserved $T^{\mu\nu}$

First we claim that Lorentz invariance implies:

$$\langle k, \sigma| J^{\mu} |k', \sigma\rangle \xrightarrow{k \to k'} \frac{q k^{\mu}}{k^0} \frac{1}{(2\pi)^3} \tag{1.6}$$

$$\langle k, \sigma| T^{\mu\nu} |k', \sigma\rangle \xrightarrow{k \to k'} \frac{k^{\mu} k^{\nu}}{k^0} \frac{1}{(2\pi)^3} \tag{1.7}$$

where $\langle k, \sigma \,|\, k', \sigma' \rangle = \delta_{\sigma\sigma'} \delta^{(3)}(\mathbf{k} - \mathbf{k}')$. When looking at 0th component of eqn(1.6), we have $\langle k, \sigma \,| J^0 |\, k', \sigma\rangle \xrightarrow{k \to k'} \frac{q}{(2\pi)^3}$.

For massless particle $k^2 = k'^2 = 0$. This implies that $k^{\mu} k'_{\mu} < 0$, i.e. $k + k'$ is timelike. We can choose a frame , such that $\mathbf{k} + \mathbf{k}' = 0$ and $k^{\mu} = (E, 0, 0, E)$ and $k'^{\mu} = (E, 0, 0, -E)$. In this frame a rotation by θ around z axis has the effect :

$$\hat{R}(\theta)\,|k,j\rangle = e^{ij\theta}\,|k,j\rangle\,, \quad \hat{R}(\theta)\,|k',j\rangle = e^{-ij\theta}\,|k',j\rangle \tag{1.8}$$

$$\langle k',j\left|\hat{R}^{-1}(\theta)J^{\mu}\hat{R}(\theta)\right|k,j\rangle = e^{2ij\theta}\langle k',j\,|J^{\mu}|\,k,j\rangle \tag{1.9}$$

$$\Rightarrow e^{2ij\theta}\langle k',j\,|J^{\mu}|\,k,j\rangle = \Lambda^{\mu}_{\nu}(\theta)\langle k',j\,|J^{\nu}|\,k,j\rangle \tag{1.10}$$

here in eqn(1.8) minus sign occurs because k' has the opposite orientation as that of k so helicity should also changes sign. Λ^{μ}_{ν} is defined by the rotation acting on a 4-vector by an angle θ around z-axis. Similarly,

$$e^{2ij\theta}\langle k',j\,|T^{\mu\nu}|\,k,j\rangle = \Lambda^{\mu}_{\rho}(\theta)\Lambda^{\nu}_{\lambda}(\theta)\langle k',j\left|T^{\rho\lambda}\right|k,j\rangle \tag{1.11}$$

Thus $\langle k',j\left|\hat{R}^{-1}(\theta)J^{\mu}\hat{R}(\theta)\right|k,j\rangle$ can only be nonzero if $j \leq 1/2$. Otherwise eqn(1.6) is contradicted because $\Lambda^{\mu}_{\nu}(\theta)$ has the eigen values $e^{\pm i\theta}, 1$. $\langle k',j\,|T^{\mu\nu}|\,k,j\rangle$ can only be nonzero if $j \leq 1$. Otherwise eqn(1.7) is contradicted. Thus we proved both the theorems. □

Weinberg-Witten Theorem forbids the existence of massless spin-2 particles, which is a hall mark of gravity, in the same space time a QFT lives. But there is a loophole: emergent gravity can live in a different spacetime, as in holographic duality.

Remarks:

1. Blackhole thermodynamics ⇒ holographic principle.
2. Large N gauge theories ⇒ gauge/string duality.

1.2 Black Holes

Let us first compare the strength of gravity and strength of electro-magnetic (EM) interaction. In the EM case, interaction takes the form $V_{EM} = e^2/r$. We take the reduced Compton wavelength $r_c = \frac{\hbar}{mc}$ to be the smallest distance between particles, because this distance can be thought as the fundamental limitation on measuring the positions of a particle, taking quantum mechanics and special relativity into account. Using the unit of particle static mass, the EM interaction has the effective strength:

$$\lambda_{EM} = \frac{V_{EM}(r_C)}{mc^2} = \frac{e^2}{\hbar c} = \alpha = \frac{1}{137} \tag{1.12}$$

On the other hand, we can also get the effective strength of gravity:

$$\lambda_G = \frac{V_G(r_c)}{mc^2} = \frac{G_N m^2}{\hbar/mc}\frac{1}{mc^2} = \frac{m^2}{m_p^2} = \frac{l_p^2}{r_c^2} \tag{1.13}$$

Then $\lambda_G \ll 1$ for $m \ll m_p$ where m_p = Planck mass and l_p = Planck length. For example, in the case of electron , $m_e = 5 \times 10^{-4} GeV/c^2$, we have

$$\frac{\lambda_G}{\lambda_{EM}} \approx 10^{-43} \tag{1.14}$$

The gravity effect is quite weak in this case. But if the mass is at Planck mass scale m_p, then $\lambda_G \sim \mathcal{O}(1)$, which means quantum gravity effects become significant (the corresponding length scale will be l_p).

1.2.1 Schwarzschild Radius

For an object of mass m, at what distance r_s from it, the classical gravity becomes strong ? To answer this question, we can consider a probe mass m', then the classical gravity becomes strong, means that

$$\frac{G_N m m'/r_s}{m'c^2} \sim 1 \quad \Rightarrow r_s = \frac{G_N m}{c^2} \tag{1.15}$$

So now for an object of mass m, we have two important scales:

1. $r_c = \frac{\hbar}{mc} \Rightarrow$ Reduced Compton wavelength.

2. $r_s = \frac{2G_N m}{c^2} \Rightarrow$ Schwarzschild radius.

The pre-factor 2 of r_s comes from a GR computation of a Schwarzschild black hole. From $\frac{r_s}{r_c} \sim \frac{m^2}{m_p^2}$, we can conclude

1. $m \gg m_p$, $r_s \gg r_c$: classical gravity (quantum effects not important);

2. $m \ll m_p$, $r_s \ll r_c$: r_s is not relevant, gravity effect is weak and not important;

3.$m \sim m_p$, $r_s \sim r_c$ quantum gravity effects are important.

If this were the whole story, life would be much simpler, but much less interesting. However, black holes can make quantum gravity effects manifest at macroscopic level, at length scales of $\mathcal{O}(r_s)$.

Remark 1.2.1. l_p can be thought as the minimal localization strength. In non-gravitational physics, the probing length scale $l \sim \frac{\hbar}{p}$, in principle, can be as small as one wants if one is powerful enough to get sufficiently large p. But with gravity, when $E \sim p \gg m_p$, then $r_s \sim \frac{G_N p}{c^3}$ takes over as the minimal scale. Since $r_s \propto p$, so larger energies give larger length scales, l_p is the minimal scale one can probe.

Alternatively, consider uncertainty principle $\delta p \sim \frac{\hbar}{\delta x}$, then $\delta x > \frac{G_N \delta p}{c^3} \sim \frac{G_N \hbar}{c^3 \delta x}$, so obtain $\delta x > \sqrt{\frac{\hbar G_N}{c^3}} = l_p$.

1.2.2 Classical Black Hole Geometry

Black hole geometry is the solution of Einstein equation with zero cosmology constant. The spacetime is due to an object of mass M. If we consider the object to be spherically symmetric, non-rotational, neutral, we have the Schwarzschild metric solution:

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \;\; f = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r} \tag{1.16}$$

The event horizon is defined at $r = r_s = 2G_N M$ where $g_{tt} = 0$, $g_{rr} = \infty$. When r goes across the event horizon, f changes sign, r and t switches their role.

Remark 1.2.2. 1. It is time-reversal invariant, i.e. invariant under $t \to -t$. It does not describe a black hole formed from gravitational collapse which is clearly not time-reversal symmetric, but it is a mathematical idealization of such a black hole.

2. The spacetime is non-singular at the horizon, as one can check this by computing curvature invariants ($I = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 48 \frac{G_N^2 M^2}{r^6}$ =Kretschmann scalar). It is only a coordinate singularity (not an intrinsic singularity), where t (Schwarzschild time) and r coordinates become singular at the horizon.

3. The horizon is a surface of infinite redshift. Consider an observer O_h at the hypersurface $r = r_h \approx r_s$ and another observer O_∞ at the hypersurface $r = \infty$. At $r = \infty : ds^2 \to -dt^2 + dr^2 + r^2 d\Omega^2$, t is the proper time for O_∞. On the other hand,

at $r = r_h : ds^2 = -f(r_h)dt^2 + \cdots = -d\tau^2 + \cdots$. We have $d\tau^h = f^{1/2}(r_h)dt$, with τ_h to be the proper time for O_h. Then

$$\frac{d\tau_h}{dt} = (1 - \frac{r_s}{r_h})^{\frac{1}{2}} \tag{1.17}$$

As $r_h \to r_s$, $\frac{d\tau_h}{dt} \to 0$, i.e. compared to the time at $r = \infty$, the time at $r = r_h$ becomes infinitely slow. Consider some event of energy E_h happening at $r = r_h$, to O_∞ this event has energy $E_\infty = E_h f^{\frac{1}{2}}(r_h)$ i.e. for fixed local proper energy E_h, $E_\infty \to 0$ as $r_h \to r_s$, we call it infinitely redshifted.

4. It takes a free-fall traveler a finite proper time to reach the horizon, but infinite Schwarzschild time.

5. Two intrinsic geometric quantities of the horizon:

- Area of a spatial section $A = 4\pi r_s^2 = 16\pi G_N^2 M^2$
- Surface gravity: The acceleration of a stationary observer at the horizon as measured by an observer at infinity is given by $K = \frac{1}{2}f'(r_s) = \frac{1}{4G_N M}$

1.2.3 Rindler Spacetime and Causal Structure

To understand the spacetime structure of a blackhole, let us consider the region near (but outside) the horizon. Introducing the proper distance ρ from the horizon:

$$d\rho = \frac{dr}{\sqrt{f}} \xrightarrow{r \to r_s} \frac{dr}{\sqrt{f'(r_s)(r - r_s) + \cdots}} \tag{1.18}$$

$$\Rightarrow \rho = \frac{2}{\sqrt{f'(r_s)}}\sqrt{r - r_s} \tag{1.19}$$

We can express it as a function of ρ

$$f(r) = f'(r_s)(r - r_s) + \cdots = (\frac{1}{2}f'(r_s))^2\rho^2 + \cdots = K^2\rho^2 + \cdots \tag{1.20}$$

Where K is the surface gravity. Near the horizon, we have

$$ds^2 = -K^2\rho^2 dt^2 + d\rho^2 + r_s^2 d\Omega_2^2 = -\rho^2 d\eta^2 + d\rho^2 + r_s^2 d\Omega_2^2 \tag{1.21}$$

Here we define $\eta = Kt = \frac{t}{2r_s}$. The first two terms in the above expression is called (1+1)d Minkowski metric in a Rindler form.

Consider $\mathcal{M}_2$ (2d Minkowski spacetime):

$$ds^2_{\mathcal{M}_2} = -dT^2 + dX^2 \tag{1.22}$$

Let $X = \rho\cosh\eta$, $T = \rho\sinh\eta$, then

$$ds^2_{\mathcal{M}_2} = -\rho^2 d\eta^2 + d\rho^2 \tag{1.23}$$

But since $X^2 - T^2 = \rho^2 \geq 0$, (ρ, η) coordinates only covers a part of $\mathcal{M}_2$. And $\rho \geq 0$ sector corresponds to $X \geq 0$ i.e. region I as shown in Fig 1.1.

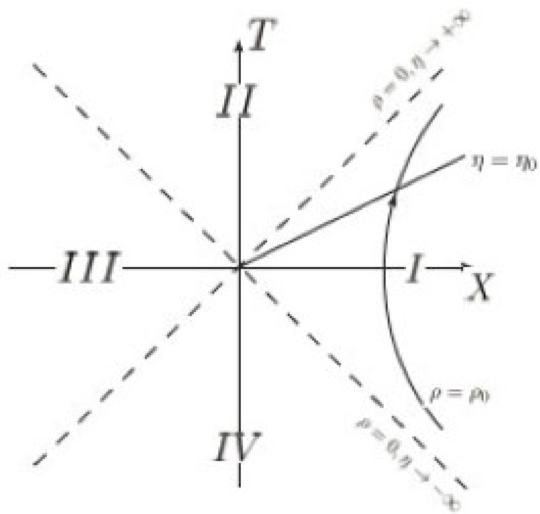


Figure 1.1: Causal structure of $\mathcal{M}_2$ in the Rindler form.

Note that,

1. $X = T(X > 0) : \eta \to \infty, \rho \to 0$ with ρe^{η} finite
2. $X = -T(X > 0) : \eta \to -\infty, \rho \to 0$ with $\rho e^{-\eta}$ finite
3. $X = T =) : \rho \to 0$, any finite η

Thus the horizon of a black hole $\rho = 0$ is mapped to a light cone $X = \pm T$. And near horizon black hole geometry can be viewed as Rindler $\times \mathcal{S}^2$ as shown in fig 1.2

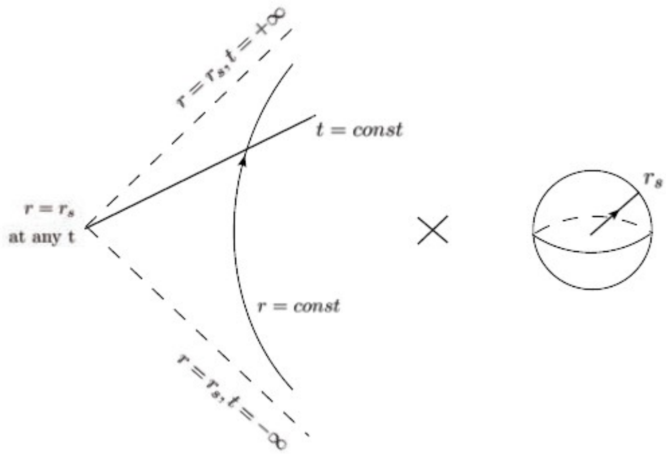


Figure 1.2: Black hole geometry near horizon.

Remark 1.2.3. 1. An observer at $r = const(r \geq r_s)$ is mapped to and observer with $\rho = const$ in a Rindler patch, i.e. an observer in Minkowski space-time following a hyperbolic trajectory $X^2 - T^2 = \rho^2 = const$. Such an observer has a proper acceleration

$$a = \frac{1}{\rho} = \frac{f'(r_s)}{2} \frac{1}{\sqrt{r - r_s}} \tag{1.24}$$

And furthermore, the acceleration seen by O_∞ would be $a_\infty = a(r)f^{1/2}(r) = K$.

2. A free-fall observer near a black hole horizon is equal to an inertial observer in $\mathcal{M}_2$.

3. Rindler coordinates(ρ, η) become singular at $\rho = 0$, but using Minkowski coordinates (X,T), one could extent region I to the full Minkowski spacetime. Similarly, by changing to suitable coordinates (Kruskal coordinates), one can extend the Schwarzschild spacetime to four patches (Fig.1.3).

- Clearly, no information or observer in region II can reach region I (separated by a future horizon).

- Region III and IV are related to I and II by time reversal. They do not exist for real blackholes formed from gravitational collapse. Observer in region I cannot influence events in region IV (separated by a past horizon).
- At r=0, there is a black hole singularity, called curvature singularity which is space-like.

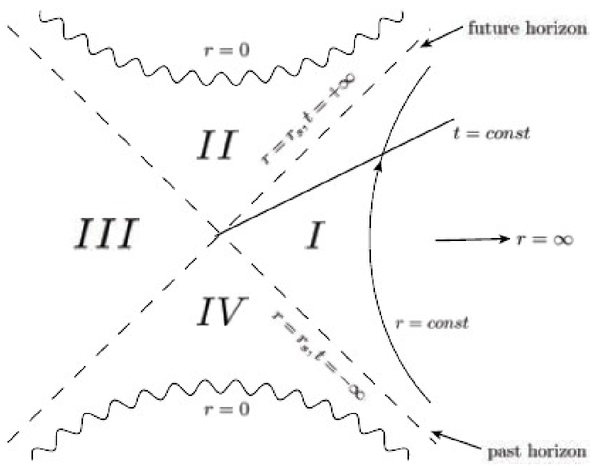


Figure 1.3: Schwarzschild black hole geometry in Kruskal coordinates.

1.2.4 Penrose Diagram

In this section, we study Penrose diagrams, which are used to visualize the global causal structure of a spacetime. We start with the metric: $ds^2 = g_{ab}(x)dx^a dx^b$

1. Find a coordinate transformation $x^a = x^a(y^\alpha)$ so that y^α has a finite range (map the whole spacetime to a finite region).

2. Construct a new metric which is conformally related to the original one

$$d\tilde{s}^2 = \Omega^2(y) ds^2 = \tilde{g}_{\alpha\beta}(y) dy^\alpha dy^\beta \tag{1.25}$$

Such that $\tilde{g}_{\alpha\beta}$ is simple. $d\tilde{s}^2$ and ds^2 have the same causal structure as null rays are preserved by conformal scalings.

Example : (1+1)d Minkowski space

$$ds^2 = -dT^2 + dX^2 = -dUdV; \quad U = T - X, V = T + X$$

let $U = \tan u, V = \tan v$, then $u, v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We define the following:

- i_0: spatial infinity (X infinite, T finite)
- i_+: time-like future infinity ($T \to \infty$, X finite)
- i_-: time-like past infinity ($T \to -\infty$, X finite)
- I_+: null future infinity (where all null rays end)
- I_-: null past infinity (where all null rays start)

Label these points (lines) accordingly in the Penrose diagram for $\mathcal{M}_2$ in fig. 1.4

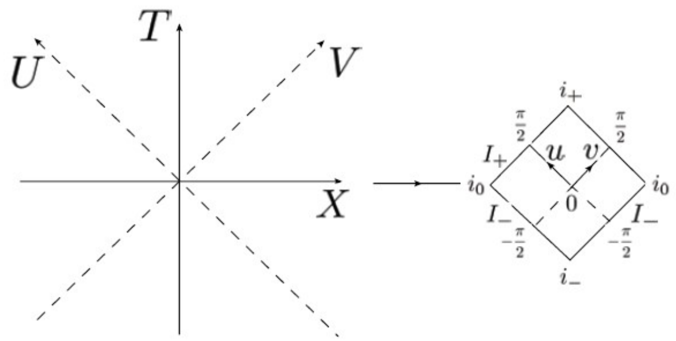


Figure 1.4: $\mathcal{M}_2$ Penrose diagram.

Another more interesting example is the Schwarzschild black hole

- We first consider (r, t) plane
- Then we go to a coordinate system (Kruskal) which covers all four regions (analogue of U, V in Minkowski spacetime).
- Next we make a coordinate transformation to make the new coordinate with a finite range $(U, V) \to (u, v)$

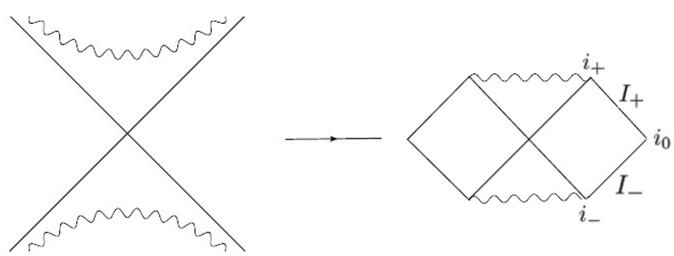


Figure 1.5: $\mathcal{M}_2$ Schwarzschild black hole Penrose diagram.

1.2.5 Black Hole Temperature

In QFT, to describe a system at finite temperature (T), we analytically continue to Euclidean signature, i.e. $t \to -\tau$. And let τ to be periodic: $\tau \sim \tau + \hbar\beta$, with $\beta = \frac{1}{T}$. If we analytically continue the Schwarzschild metric to Euclidean signature with $t \to -i\tau$, near horizon we get

$$dS_E^2 = \rho^2 K^2 d\tau^2 + d\rho^2 + r^2 d\Omega_2^2 = d\rho^2 + r_s^2 d\Omega_2^2; \quad \theta = K\tau = \frac{\tau}{2r_s} \tag{1.26}$$

Note that the first two terms above describe a polar coordinates in Euclidean $\mathcal{R}^2$. This metric has a conical singularity unless θ is periodic in 2π, i.e. $\theta \sim \theta + 2\pi$. Since horizon is non-singular in Lorentzian signature, it should not be singular in Euclidean. Hence τ must be periodic

$$\tau \sim \tau + \frac{2\pi}{K} \tag{1.27}$$

Recall that t is the proper time for an observer at $r = \infty$, an observer at $r = \infty$ must feel a temperature :

$$T = \frac{1}{\beta} = \frac{\hbar K}{2\pi} = \frac{\hbar}{8\pi G_N m} \tag{1.28}$$

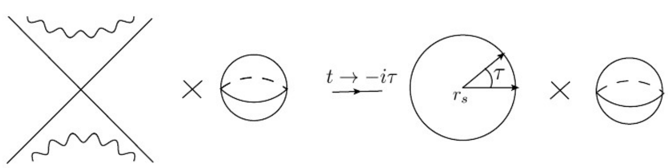


Figure 1.6: Schwarzschild black hole near horizon geometry in Minkowski (left) and Euclidean (right) signature.

For an observer at some r, since $dt_{loc} = f^{\frac{1}{2}}(r)dt$, we have This local temperature goes to ∞ as we approach the horizon, i.e. the black hole horizon is a very hot place for a stationary observer!

Similarly for Rindler spacetime

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \rightarrow \eta = -i\theta ds_E^2 = \rho^2 d\theta^2 + d\rho^2 \tag{1.29}$$

Since θ must be periodic in 2π, we define the local proper time: $d\tau_{loc}^2 = \rho^2 d\theta^2$. Then τ_{loc} must be periodic

$$T_{loc}^{Rindler}(\rho) = \frac{\hbar}{2\pi\rho} = \frac{\hbar a}{2\pi} \tag{1.30}$$

where $a = \frac{1}{\rho}$. So in Minkowski spacetime, an accelerated observer will feel a temperature proportional to its acceleration!

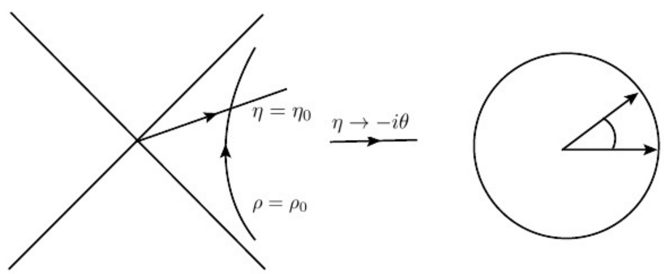


Figure 1.7: Rindler spacetime in Minkowski (left) and Euclidean (right) signature.

Physical interpretation of the temperature

Consider a QFT in a black hole spacetime. The "vacuum" state obtained via this analytic continuation procedure from the Euclidean signature is a thermal equilibrium state with the stated temperature.

Remark 1.2.4. 1. The choice of vacuum for a QFT in a curved spacetime is not unique. In the Schwarzschild black hole case, it is the "Hartle-Hawking vacuum"; while in the Rindler case, it is the Minkowski vacuum reduced to the Rindler patch (reduced density matrix of the Minkowski vacuum).

2. If for a black hole, in Euclidean signature we take τ to be noncompact, then it is the Schwarzschild vacuum (Boulware vacuum). This is the vacuum that one would get by doing canonical quantization in terms of the Schwarzschild time t. In the Rindler case, if we take θ to be noncompact, we have Rindler vacuum, which can be obtained by doing canonical quantization in Rindler patch in terms of η.

3. In the Schwarzschild vacuum, since the correspoding Euclidean manifold is singular at the horizon, physical observables are often singular there, *e.g.* stress tensor blows up there. But in Lorentzian signature, this is not the case.

1.2.6 Black Hole Thermodynamics

From the previous discussion, we know that a black hole has a temperature:

$$T_{BH} = \frac{\hbar}{8\pi G_N m}$$

Thus a black hole is a thermodynamic object, and it must obey thermodynamics. Now recall thermodynamic relations:

$$\frac{dS}{dE} = \frac{1}{T(E)} = \frac{8\pi G_N m}{\hbar} \tag{1.31}$$

since for a black hole E = m, Entropy

$$S(E) = \int \frac{dE}{T(E)} = \frac{4\pi G_N E^2}{\hbar} + const = \frac{4\pi r_s^2}{4\hbar G_N} = \frac{A_{BH}}{e\hbar G_N} \tag{1.32}$$

The integral constant can be determined to be 0 since S(E)=0 for E=0, A_{BH} is the area of black hole horizon. So we now have the most important conclusion for black hole

$$T_{BH} = \frac{\hbar K}{2\pi}, \quad S_{BH} = \frac{A_{BH}}{4\hbar G_N} \tag{1.33}$$

Note that T_{BH} decreases as mass m increases, the system has a negative specific heat:

$$C = T\frac{\partial S}{\partial T} = \frac{\partial E}{\partial T} < 0 \tag{1.34}$$

General Black Holes

Theorem 1.2.5. ***No hair theorem***:*A stationary, asymptotically flat black hole is characterized by its mass M, angular momentum J, conserved gauged charges (e.g electric charges Q).*

Now we summarize four laws of black hole mechanics:

- 0th law: Surface gravity K is constant over the horizon.
- 1st law:

$$dM = \frac{K}{8\pi G_N} dA + \Omega dJ + \phi dQ \tag{1.35}$$

$$\Rightarrow dE = TdS + \Omega dJ + \phi dQ \tag{1.36}$$

where Ω is the angular frequency at the horizon, ϕ is the electric potential at the horizon (assume that at ∞ the potential is 0.)

- 2nd law: Horizon area never decreases classically.
- 3rd law: Surface gravity of a black hole cannot be reduced to 0 in a finite number of steps.

Historically, before Hawking's discovery of black hole radiation, Bekenstein(1972-1974) has found $S_{BH} \sim A_H$,the motivation is to save the 2nd law of thermodynamics for a system with blackholes. If an ordinary system falls into a blackhole, the ordinary entropy becomes invisible to an exterior observer, therefore we have the generalized 2nd law(GSL):

$$dS_{tot} \geq 0; \quad S_{tot} = S_{BH} + S_{matter}$$

Consequently, we get some puzzles/paradoxes

1. Does black hole entropy has a statistical interpretation ?
2. Does black hole respect quantum mechanics ?

The first question has been answered in the affirmative for many different types of black holes in string theory and holographic duality. That is a black hole has internal states of order:

$$N \sim e^{\frac{A_{BH}}{4\hbar G_N}} \tag{1.37}$$

The second question is related to Hawking's information loss paradox. The rough description of this paradox is: consider a star in a pure state collapse to form a black hole, which then radiates thermally. If to a good approximation, the radiation is thermal for $m \gg m_p$, so before $m \sim \mathcal{O}(m_p)$, very little information about the original state can come out. Once $m \sim \mathcal{O}(m_p)$, it will be too late for all the information to go out. Then we start from a pure state and eventually get into a thermal state with density matrix description, i.e. information is lost!

Chapter 2

Holographic Duality

2.1 Holographic Principle

If we do treat black hole as an "ordinary" QM object, an important implication would be the holographic principle.

Consider an isolated system of mass E and entropy S_0 in an asymptotic flat space-time. Let A be the area of the smallest sphere that encompasses the system, and M_A to be the mass of a black hole with the same horizon area, we must have $E < M_A$, otherwise the system would be already a black hole.

Now add $M_A - E$ energy to the system (keeping A fixed), we shall obtain a black hole with mass M_A,since

$$S_{BH} \geq S_0 + S' \tag{2.1}$$

where S' is the entropy of added energy, we conclude that

$$S_0 \leq S_{BH} = \frac{A}{4\hbar G_N} \tag{2.2}$$

i.e. the maximal entropy inside a region bounded by area A is,

$$S_{max} = \frac{A}{4\hbar G_N} \tag{2.3}$$

Recall the definition of entropy in quantum statistical physics: $S = -Tr\rho log\rho$, where ρ is the density matrix for the state of a system. For a system with N-dimensional Hilbert space

$$S_{max} = logN \tag{2.4}$$

Hence the effective dimension of Hilbert space for system inside a region of area A is bounded by,

$$logN \leq \frac{A}{4\hbar G_N} = \frac{A}{4l_p^2} \tag{2.5}$$

Holographic Principle: In quantum gravity, a regime of boundary area A can be fully described by no more than $\frac{A}{4\hbar G_N} = \frac{A}{4l_p^2}$ d.o.f.

2.2 Large N Expansion of Gauge Theories

We now look at clues to holographic duality from field theory side. Consider QCD which can be described as SU(3) gauge theory with fundamental quarks. The Lagrangian reads

$$\mathcal{L} = \frac{1}{g_{YM}^2}\left[-\frac{1}{4}TrF_{\mu\nu}F^{\mu\nu} - i\overline{\Psi}(\not{D} - m)\Psi\right] \tag{2.6}$$

where $D_\mu = \partial_\mu - iA_\mu$, A_μ are 3×3 Hermitian matrices and can be expressed as $A_\mu = A^a_\mu T^a$, with $T^a \in SU(3)$ In such a theory, coupling becomes strong in IR ($\Lambda_{QCD} \sim 250$ MeV), there is no small parameter to expand. It is still an open problem to derive IR properties of QCD from the first principle.

t' Hooft in 1974 suggested to take the number of colors N = 3 as a parameter, i.e. promote A_μ to $N \times N$ hermitian matrices and consider $N \to \infty$ and do $\frac{1}{N}$ expansion. It is an ingenious idea. Unfortunately, QCD still can not be solved to leading order in the large N limit. Surprisingly, there is an correspondence between the large N gauge theory and the string theory. The key is that fields are matrices. As an illustration, we will consider a scalar theory:

$$\mathcal{L} = -\frac{1}{g^2} Tr\left[\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{4}\Phi^4\right] \tag{2.7}$$

where g is the coupling constant, $\Phi(x) \equiv \Phi^a_b(x)$: $N \times N$ hermitian matrix. In terms of components

$$\mathcal{L} = -\frac{1}{g^2}\left[\frac{1}{2}(\partial_\mu \Phi^a_b)(\partial^\mu \Phi^b_a) + \frac{1}{4}\Phi^a_b \Phi^b_c \Phi^c_d \Phi^d_a\right] \tag{2.8}$$

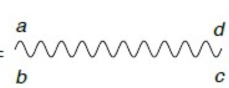
$\mathcal{L}$ is invariant under $\mathcal{U}(N)$ global symmetry.

Remark 2.2.1. 1. It is a theory of N^2 scalar fields.

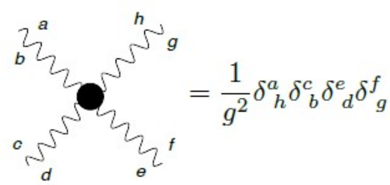
2. One can also consider other types of matrices, $e.g. N \times N$ real symmetric matrix, the corresponding symmetry will be SO(N).

3. One could also introduce gauge fields to make U(N) symmetry local.

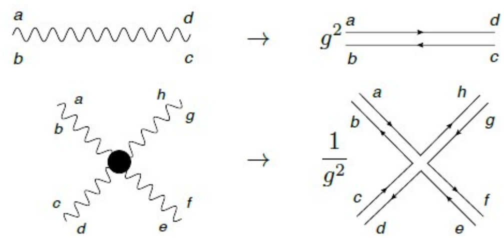
Here we list the Feynman rules for this theory: The propagator:

$\langle \Phi^a{}_b(x) \Phi^c{}_d(y) \rangle =$  $= g^2 \delta^a{}_d \delta^c{}_b G(x-y)$

The fermion vertex:

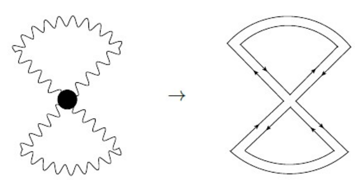


So here we can adapt the double line notation:

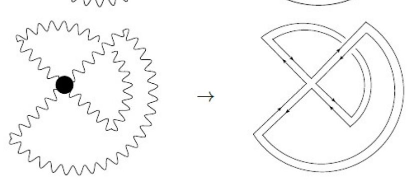


2.2.1 Vaccum Energy

We consider vacuum bubbles, i.e. diagrams with no external legs. The lowest order diagrams will be



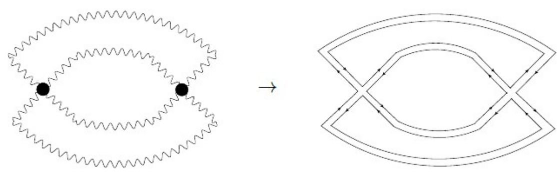
(1)



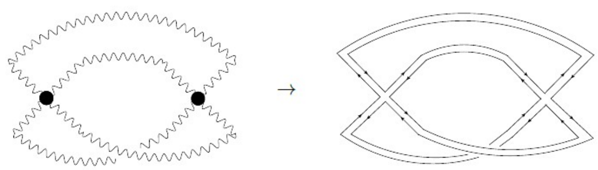
(2)

In the case of diagram 1, each contracted index line gives N, so the total contribution will be of the order $N^3 \frac{(g^2)^2}{g^2} = N^3 g^2$ In the case of diagram 2, there is only one contracted line, the total contribution will be of the order Ng^2. The difference comes from the fact that the matrices do not commute. In the first case, the diagram can be drawn on a plane without crossing lines, we call it a plannar diagram; while in the second case, the diagram cannot be drawn on a plane without crossing lines, we call it a non-planar diagram.

If we consider next order in the perturbation theory



(3)

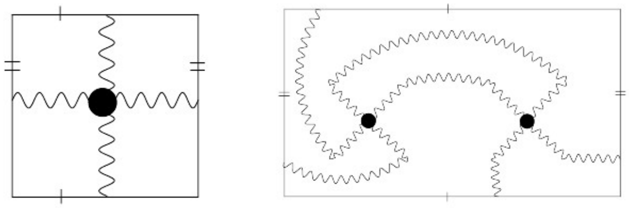


(4)

The first diagram gives the order of $N^4 g^4$, the second diagram gives the order of $N^2 g^4$. We can further consider higher order diagrams, but how can we obtain general N-counting? And how to classify all the non-planar diagrams ?

To answer the above questions, we make 2 observations

- Diagrams 2 and 4 can be drawn on a torus without crossing lines.

 (5)

- The power of N for each diagram equals to the number of faces in the diagram after we straighten it out.

In fact, any orientable two dimensional surface is classified topologically by an integer h, called the genus. The genus is equal to the number of "holes" that the surface has

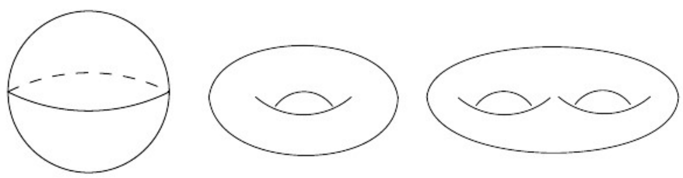


Figure 2.1: sphere (genus-0), torus (genus-1) and double torus(genus-2).

An topological invariant of the manifold is the Euler character:

$$\chi = 2 - 2h \tag{2.9}$$

1. For any non-planar diagram, there exists an integer h, such that the diagram can be straightened out (i.e. non-crossing) on a genus-h surface, but not on a surface with a smaller genus.

2. For any non-planar diagram, the power of N that comes from contracting propagators is given by the number of faces on such a genus-h surface, i.e. the number of disconnected regions separated by the diagram.

In general, a vacuum diagram has the following dependence on g^2 and N:

$$A \sim (g^2)^E (g^2)^{-V} N^F \tag{2.10}$$

where E is the number of propagators, V is the number of vertices, F is the number of faces. This does not give a sensible $N \to \infty$ limit or $\frac{1}{N}$ expansion, since there is no upper limit on F. However, 't Hooft suggests that we can take the limit $N \to \infty$ $g^2 \to 0$ but keep $\lambda = g^2 N$ fixed. Then if L is the number of loops,

$$A \sim (g^2 N)^{E-V} N^{F+V-E} = \lambda^{L-1} N^{\chi} \tag{2.11}$$

Theorem 2.2.2. *Given a surface composed of polygons with F faces, E edges and V vertices, the Euler character satisfy*

$$\chi = F + V - E = 2 - 2h \tag{2.12}$$

Since each Feynman diagram can be considered as a partition of the surface separating it into polygons, then the above theorem also works for our counting in N.

Thus in this limit, to the leading order in N are the planar diagrams

$$N^2(c_0 + c_1\lambda + c_2\lambda^2 + \cdots) = N^2 f_0(\lambda) \tag{2.13}$$

Because $logZ$ evaluates the sum of all vacuum diagrams, we can conclude, including higher order $1/N^2$ corrections:

$$logZ = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda) = N^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \cdots \tag{2.14}$$

The first term comes from the planar diagrams, second term from the genus-1 diagrams, etc.

There is a heuristic way to understand $logZ = \mathcal{O}(N^2) + \cdots$. Since $Z = \int \mathcal{D}\Phi e^{iS[\Phi]}$ and we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{N}{\lambda} Tr \left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}\Phi^4 \right] \tag{2.15}$$

The trace also gives a factor of N, thus $\mathcal{L} \sim \mathcal{O}(N^2)$ and we have $logZ \sim \mathcal{O}(N^2)$.

2.2.2 General Observables

Consider allowed operators in the two theories. In eq.(2.7), operators like Φ^a_b are allowed, although it is not invariant under global U(N) symmetry. But in eq.(2.6), allowed operators must be gauge invariant, so Φ^a_b is not allowed. So if we consider gauge theories: $\mathcal{L} = \mathcal{L}(A_\mu, \Phi, \cdots)$, the allowed operators will be

1. Single-trace operators : $Tr(F_{\mu\nu}F^{\mu\nu}), Tr(\Phi^n)$
2. Multiple-trace operators : $Tr(F_{\mu\nu}F^{\mu\nu})Tr(\Phi^2), Tr(\Phi^2)Tr(\Phi^n)Tr(\Phi^n), \cdots$

We denote single-trace operators as $\mathcal{O}_k, k = 1; \cdots$ represents different operators. Then multiple-trace ones will be like $\mathcal{O}_m\mathcal{O}_n(x), \mathcal{O}_{m_1}\mathcal{O}_{m_2}\mathcal{O}_{m_3}(x), \cdots$

So general observables will be correlation functions of gauge invariant operators, here we focus on local operators :

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c \tag{2.16}$$

Since we are working in the t'Hooft limit, we want to know how correlation (Eq.2.16) scales in the large N limit. There is a trick, consider

$$Z\left[J_1,\cdots,J_n\right] = \int \mathcal{D}A_\mu \mathcal{D}\Phi\cdots e^{iS_{eff}} = \int \mathcal{D}A_\mu \mathcal{D}\Phi\cdots e^{\left[iS_0 + iN\sum_i \int J_i(x)\mathcal{O}_i(x)\right]} \tag{2.17}$$

Then the correlation (Eq. 2.16) can be expressed as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c = \frac{\delta^n logZ}{\delta J_1(x_1)\cdots\delta J_n(x_n)}\Big|_{J_1=\cdots=J_n=0}\frac{1}{(iN)^n} \tag{2.18}$$

Applying eq. 2.18 on eq. 2.14 we get,

$$\langle \mathbb{I}\rangle \sim \mathcal{O}(N^2) + \mathcal{O}(N^0) + \cdots \tag{2.19}$$

$$\langle \mathcal{O}\rangle \sim \mathcal{O}(N) + \mathcal{O}(N^{-1}) + \cdots \tag{2.20}$$

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle_c \sim \mathcal{O}(N^0) + \mathcal{O}(N^{-2}) + \cdots \tag{2.21}$$

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_c \sim \mathcal{O}(N^{-1}) + \mathcal{O}(N^{-3}) + \cdots \tag{2.22}$$

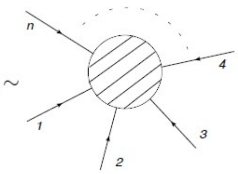
All leading order contributions come from planar diagrams.

Physical Implications :

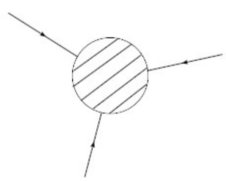
1. In the large N limit, $\mathcal{O}(x)|0\rangle$ can be interpreted as creating a single-particle state("glue ball"). Similarly $:\mathcal{O}_1\cdots\mathcal{O}_n(x):|0\rangle$ represents n-particle state.

2. The fluctuations of "glue balls" are suppressed.

3. If we interpret as "scattering amplitude" of n "glue balls", then to the leading order in $N \to \infty$, the scattering only involve tree-level interactions (only classical), among the glue ball states.

$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c \sim$  $\sim O(N^{2-n}) + \cdots$

   (a) Consider

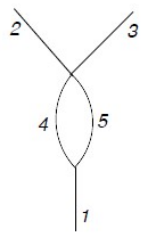
 $\sim \frac{1}{N} \sim \tilde{g}$

   suppose we treat it as a basic vertex with coupling $\tilde{g}$, then the tree-level amplitude for n-particle scattering scales as $\tilde{g}^{n-2} \sim N^{2-n}$.

   (b) We can also include higher order vertices, but they should satisfy:

 $\sim \tilde{g}^2$,  $\sim \tilde{g}^3, \quad \cdots$

   (c) There are no more than one-particle intermediate states. If we insert a complete set of states at all possible places, due to large N counting, all states other than-single particle ones are suppressed:



$$= \langle \mathcal{O}_1 : \mathcal{O}_4\mathcal{O}_5 :\rangle\langle : \mathcal{O}_4\mathcal{O}_5 : \mathcal{O}_2\mathcal{O}_3\rangle \sim O(N^{-3})$$

Compared to

$$= \langle \mathcal{O}_1\mathcal{O}_i\rangle\langle \mathcal{O}_i\mathcal{O}_2\mathcal{O}_3\rangle \sim O(N^{-1})$$

i.e. all "loops" of glue balls are suppressed.

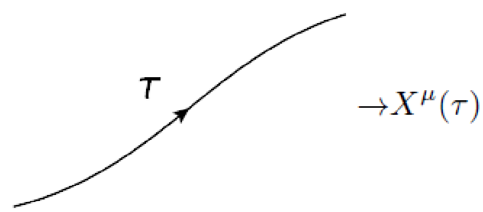
More explicitly

Gauge theory with finite $\hbar$ in the $N \to \infty$ limit = Glue ball theory
Perturbative expansion in 1/N = Loops of glue balls perturbative in $\hbar$

2.3 Large N Expansion As a String Theory

QFT can be considered as a theory of "particles". The standard quantization approach is second quantization. In the first quantization approach, we directly quantize the motion of a particle in space time. We have

$$Z = \int DX^\mu(\tau) e^{iS_{particle}} \tag{2.23}$$



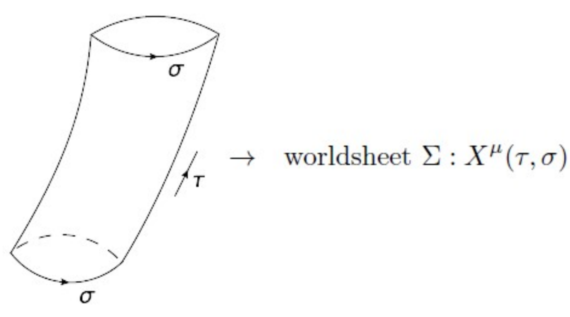
Where,

$$S_{particle} = m\int dl = m\int d\tau \frac{dl}{d\tau} = m\int d\tau \sqrt{g_{\mu\nu}\frac{dX^{\mu}}{d\tau}\frac{dX^{\nu}}{d\tau}} \tag{2.24}$$

If we want to include interactions like $\lambda\phi^3$, we need to add them by hand.

2.3.1 String Theory

In string theory, similarly, we need to quantize the motions of strings in space-time.



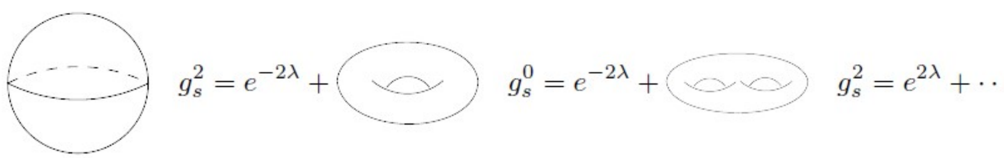
The simplest form of S_{string} is the Nambu-Goto action

$$S_{NG} = T\int_{\Sigma} dA \tag{2.25}$$

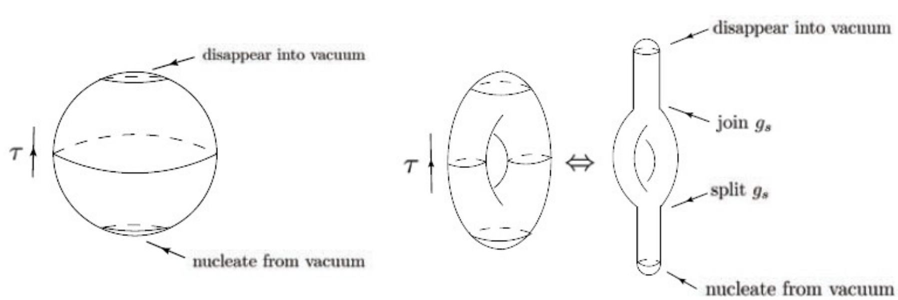
Here, $T = \frac{1}{2\pi\alpha'}$ is the string tension(mass per unit length). $dA = \sqrt{-det(h_{ab})}d\sigma d\tau$ is the infinitesimal area of the world sheet with the induced matrix $h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ To define and evaluate eq.[(2.3)], the most convenient way is to go to Euclidean signature. For vacuum processes:

$$Z_{string} = \sum_{\text{all closed surfaces}} e^{-S_{NG}} = \sum_{h=0}^{\infty} e^{-\lambda\chi} \sum_{\text{surfaces with given topology}} e^{-S_{NG}} \qquad (2.26)$$

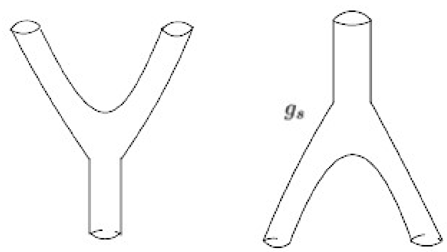
here $\chi = 2 - 2h$ denotes the weight for different topologies, λ can be thought as the "chemical potential" for topology. If we define $g_s = e^\lambda$, the vacuum includes diagrams like



There is a remarkable fact about string theory: summing over topology of all surfaces automatically includes interactions of strings. In fact this fully specifies string interactions with no freedom of making arbitrary choices. To see this,



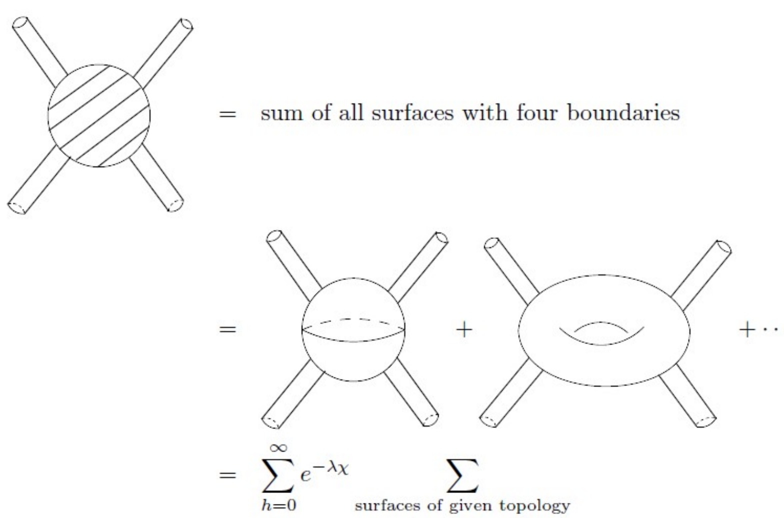
The surface can be thought as the vacuum bubble, at the south pole the string nucleates from vacuum and at the north pole, the string disappears into the vacuum. The torus can be thought as the one loop diagram, the string split into two strings and then join together again with interaction strength g_s on each vertex. Thus the basic interaction vertices are the splitting and rejoining of the strings, the coupling strength is $g_s = e^{\lambda}$:



Now we include external strings, e.g.

$$\text{string} + \text{string} \rightarrow \text{string} + \text{string} \tag{2.27}$$

In the diagrammatic language:



where $\chi = 2 - 2h - n$, where n is the number of boundaries (number of external strings).

Thus for n-string scattering process (including vacuum processes, i.e. n = 0)

$$\mathcal{A}_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} F_n^{(h)} = \underbrace{g_s^{n-2} F_n^{(0)}}_{\substack{\text{tree-level diagrams} \\ \text{(sphere topology)}}} + \underbrace{g_s^{n} F_n^{(1)}}_{\substack{\text{1-loop diagrams} \\ \text{(torus topology)}}} + \underbrace{g_s^{n+2} F_n^{(2)}}_{\substack{\text{2-loop diagrams} \\ \text{(double torus topology)}}} \tag{2.28}$$

Now comparing with the large N expansion of a gauge theory as we discussed earlier (including n=0)

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c = \sum_{h=0}^{\infty} N^{2-n-2h} f_n^{(h)} = \underbrace{N^{2-n} f_n^{(0)}}_{\substack{\text{planar diagrams} \\ \text{(sphere topology)}}} + \underbrace{N^{-n} f_n^{(1)}}_{\text{torus diagrams}} + \underbrace{N^{-n-2} f_n^{(2)}}_{\substack{\text{double-torus} \\ \text{diagrams}}} \tag{2.29}$$

Comparing eq.[(2.28)] and eq.[(2.29)], we see an identical mathematical structure of the two theories with the identification:

$e^{\lambda} = g_s \leftrightarrow \frac{1}{N}$
external strings $\leftrightarrow$ "glue balls" (single trace operator) $\mathcal{O}_i(x)\,\vert 0\rangle$
sum over string world sheet of given topology $\leftrightarrow$ sum over Feynman diagrams of given topology
topology of the world sheet $\leftrightarrow$ topology of Feynman diagrams

Recall that each Feynman diagram can be considered as a partition of a genus-h surface. The scattering amplitude of n particles on genus-h surface can be written as,

$$f_n^{(h} = \sum_{\text{all Feynman diagrams of genus-h}} G = \sum_{\text{all possible triangulations of genus-h surface}} G \tag{2.30}$$

Here G represents the expression for each diagram. Similarly in string theory, we have n-string scattering process,

$$F_n^{(h} = \int_{\substack{\text{genus h surfaces} \\ \text{with n boundaries}}} DXe^{-S_{string}} = \sum_{\substack{\text{all possible triangulations of} \\ \text{genus-h surfaces with n boundaries}}} e^{-S_{string}} \tag{2.31}$$

If we could identify G with some $e^{-S_{string}}$, we will then have:

a large N gauge theory = a string theory
$\frac{1}{N}$ expansion = perturbative expansion in g_s
large N limit (classical theory of glue-balls) = classical string theory
single-trace operators (glueballs) = string states

In fact this identification is difficult:

- String theory is formulated in the continuum, while the Feynman diagrams at best has a discrete version (triangulation of the manifold).
- The action S_{string} gives a map from the world sheet Σ to the target space $\mathcal{M}$ (space time manifold) $(\sigma, \tau) \to X^\mu(\sigma, \tau)$. In such a map, we can make choices of space time manifold $\mathcal{M}$, the specific forms of the action S_{string}, we can also have "internal" d.o.f. living on the world sheet with no immediate space-time. For example, it can be super strings, including fermions on the world sheet.

Generalizations:

1. We have so far been restricted to matrix-valued fields, i.e. fields in the adjoint representation of U(N) gauge group. One could also include fields in the

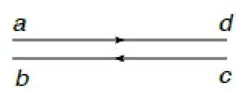
fundamental representation (quarks)

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix}$$

e.g., vacuum diagrams now include loops of quarks, which can be classified topologically by 2d surfaces with boundaries, then it corresponds to a string theory with both open and closed strings.

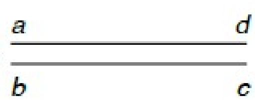
2. So far we considered U(N) gauge group,

$$\langle \Phi^a{}_b(x)\Phi^c{}_d(y)\rangle =$$



If instead, we consider SO(N) or SP(N), then there is no divergence between the two indices of the fields,

$$\langle \Phi_{ab}\Phi_{cd}\rangle =$$



Now take e.g. large N generalization of QCD in (3+1)d Minkowski spacetime. Suppose $\frac{1}{N}$ expansion can be described by a string theory, what can we say about it?

The simplest guess would be a string theory in (3+1)d Minkowski space

$$ds^2 = -dt^2 + d\vec{x}^2 \tag{2.32}$$

We can consider Nambu-Goto action or the Polyakov action which is equivalent to S_{NG} classically. But this does not work:

1. Such a string theory is inconsistent for $D \neq 10, 26$; where D is the spacetime dimension.

2. Take a string theory in 10d with $\mathcal{M}_4 \times \mathcal{N}$, where $\mathcal{N}$ is some compact manifold. Such a theory contains a mass-less spin-2 particle (graviton) in $\mathcal{M}_4$, which is not present in Yang-Mills theory.

To solve the problem, we can either think about more exotic string actions or consider other target space. Actually there are hints for considering a 5d string theory:

1. Holographic principle

   String theory necessarily contains gravity, to be consistent with holographic principle, such a gravity theory should be in 5d.

2. The consistency of string theory itself:

   It needs to include a Liouville mode which behaves as on extra dimensions.

Now consider a string Y in 5d space-time. It should at least have all the symmetries of 4d YM theories, e.g. translations, Lorentz symmetries etc. i.e. consider,

$$ds^2 = a^2(z) \left[dz^2 + \eta_{\mu\nu} dX^\mu dX^\nu\right] \tag{2.33}$$

which is the most general metric consistent with 4d Poincare symmetries. But if a theory is conformal, or simply scale invariant, Eq. [(2.33)] should be the AdS metric.

This is simple to see. If Eq.[(2.33)] is invariant under scaling transformation,

$$X^{\mu} = \lambda X^{\mu} \tag{2.34}$$

Then we must have $z \to \lambda z$ and $a(\lambda z) = \frac{1}{\lambda} a(z)$, which means $a(z) = \frac{R}{z}$ with R constant. Now as a closing touch, we make a list of the history, of the discovery of the holographic duality.

1974 (continued)

Lattice QCD (Wilson), confining strings

1993-1994

Holographic principle (t' Hooft, Susskind)

1997 June

Need 5D string theory to describe QCD (Polyakov)

1995

D-branes (Polchinski)

1997 Nov

AdS/CFT (Maldacena)

1998 Feb

Connection between holographic principle and large N gauge theory/string theory duality (Witten)

Chapter 3

Deriving AdS/CFT

3.1 Perturbative (Bosonic) String Theory

3.1.1 General Set Up

Consider a string moving in a spacetime $\mathcal{M}$ with the metric $(\mu, \nu = 0, 1, ..., d-1)$:

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu \tag{3.1}$$

With the world-sheet Σ parametrizations (a = 0, 1):

$$X^\mu(\sigma, \tau) = X^\mu(\sigma^a) \tag{3.2}$$

The induced metric on Σ is written as:

$$h_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \quad ds^2 = h_{ab}(\sigma,\tau)d\sigma^a d\sigma^b \tag{3.3}$$

The string action is defined to be proportional to the area of Σ, written in the following Nambu-Goto form:

$$S_{NG}[X^\mu] = \frac{1}{2\pi\alpha'}\int_\Sigma dA \;\; = \frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}; \quad [\alpha'] = L^2 \to \alpha' = l_s^2, \quad T = \frac{1}{2\pi\alpha'} \tag{3.4}$$

with l_s is the string length scale (from dimensional analysis) and T is the string tension. Here $d^2\sigma = d\sigma d\tau$

Since the non-polynomial nature of S_{NG} is inconvenient for calculations, it's much easier to work with the Polyakov's action, which is equivalent to the Nambu-Goto's action at classical level [1]:

$$S_P[\gamma^{ab}, X] = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}; \quad \gamma^{ab} \equiv \gamma^{ab}(\sigma,\tau) \tag{3.5}$$

Equation of Motion for γ^{ab} :

$$\gamma_{ab} = \frac{\lambda}{2}h_{ab} \tag{3.6}$$

Here, $\lambda :$ arbitrary function. So, $\gamma^{ab}h_{ab} = \frac{2}{\lambda}\times 2$ and $\sqrt{-\gamma} = \frac{\lambda}{2}\sqrt{-h}$. Eq.[(3.5)] has the form of a 2-d scalar field theory in curved space-time Σ with metric γ_{ab}. The

[1] To see this, note that world-sheet stress-energy tensor is defined as:

$$T_{ab} = -4\pi\frac{\delta S_P}{\sqrt{-\gamma}\delta\gamma^{ab}} \tag{3.7}$$

string path integral quantization will be based on

$$\int D\gamma^{ab} DX^{\mu} e^{iS_P[\gamma^{ab},X^{\mu}]} \dots \tag{3.12}$$

For the sake of understanding the physical spectrum of strings, then canonical quantization is more convenient. The Polyakov Lagrangian:

$$S_P = \int_{\Sigma} d^2\sigma \mathcal{L}_P \tag{3.13}$$

Symmetries of eq.[(3.5)] :

1. Global Poincare transformation (translation and Lorentz rotation):

$$X^{\mu}(\sigma,\tau) \to X^{\mu} + a^{\mu}; \quad X^{\mu} \to \Lambda^{\mu}_{\nu} X^{\nu} \tag{3.14}$$

Since $\delta S_P = 0$ for variations around the classical solution (on-shell), the equation of motion for γ^{ab} is $T_{ab} = 0$. Using:

$$\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{ab}\delta\gamma^{ab} \tag{3.8}$$

then the stress-energy tensor can be found:

$$\left(\int_{\Sigma} d^2\sigma\right)^{-1} \delta S_P = -\frac{1}{4\pi\alpha'}\left(\delta\sqrt{-\gamma}\gamma^{ab}h_{ab} + \sqrt{-\gamma}\delta\gamma^{ab}h_{ab}\right) = \frac{1}{4\pi\alpha'}\sqrt{-\gamma}\left(\frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} - h_{ab}\right)\delta\gamma^{ab} \tag{3.9}$$

$$\Rightarrow T_{ab} = \frac{1}{\alpha'}\left(\frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} - h_{ab}\right) = \frac{1}{\alpha'}G_{\mu\nu}\left(\frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^{\mu}\partial_d X^{\nu} - \partial_a X^{\mu}\partial_b X^{\nu}\right) = 0 \tag{3.10}$$

This means $\gamma^{ab} = Bh^{ab}$, with $B = B(\sigma,\tau)$ can be arbitrary. Integrate out the world-sheet intrinsic metric field γ^{ab}:

$$S_P\left[\gamma^{ab} = Bh^{ab}, X^{\mu}\right] = \frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\left(B^{-1}\sqrt{-h}\right)\left(Bh^{ab}\right)(h_{ab}) = \frac{1}{2\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{-h} = S_{NG}\left[X^{\mu}\right] \tag{3.11}$$

2. Local diffeomorphism transformation ($\sigma^a \to \sigma'^a$) :

$$X^\mu(\sigma,\tau) \to X'^\mu(\sigma',\tau') = X^\mu(\sigma,\tau); \quad \gamma^{ab} \to \gamma'^{ab} = \frac{\partial\sigma'^a}{\partial\sigma^c}\frac{\partial\sigma'^b}{\partial\sigma^d}\gamma^{cd}(\sigma,\tau) \qquad (3.15)$$

3. Local Weyl transformation:

$$\gamma^{ab} \to e^{-2\omega(\sigma,\tau)}\gamma^{ab}(\sigma,\tau) \qquad (3.16)$$

These symmetries (Poincare and Diff $\times$ Weyl) can be used as the guiding principles to (almost) uniquely determined the string action in eq:[(3.5)]. Indeed, for example, in a topological invariant of 2D oriented closed surfaces, the 2D Einstein-Hilbert action also fits the bill:

$$S_\chi[\gamma^{ab}] = \lambda\left(\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-\gamma}R\right) = \lambda_\chi\left(\Sigma\right); \quad \chi\left(\Sigma\right) = 2-2g \qquad (3.17)$$

Here, R = Ricci scalar.

3.2 Light-Cone Quantization

Each physical oscillation mode of a string corresponds to a particle in space-time. For mass-less mode, closed string gives a spin 2 particle (graviton) and open string gives a spin 1 particle (gauge particle, like photon or gluon).

The canonical quantization procedure:

1. Write down the classical equation of motion.

2. Fix the gauge symmetries.

3. Find the complete set of classical solution.

4. Promote classical fields (on world-sheet) to quantum operators, satisfying canonical quantization condition. The classical solutions become solutions to operator equation, and the parameters in classical solutions become creation and annihilation operators.

5. Read-of the spectrum by acting creation operators on the vacuum of the (2D world-sheet) theory.

The classical equation of motion from eq.[(3.5)]

1. For γab :
$$0 = T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu \tag{3.18}$$

2. For X^μ :
$$\partial_a \left(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu\right) = 0 \tag{3.19}$$

By diffeormophism, the metric can be put in the conformally flat form $\gamma_{ab} = e^{2\omega(\sigma,\tau)}\eta_{ab}$, and Weyl rescaling can be used to get $\gamma_{ab} = \eta_{ab}$. Now from eq.[(3.19)]

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0 \tag{3.20}$$

From eq.[(3.2)], we get,

$$T_{00} = T_{11} = \frac{1}{2}\left(\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu\right) = 0 \tag{3.21}$$
$$T_{\tau\sigma} = T_{\sigma\tau} = \partial_\tau X^\mu \partial_\sigma X_\mu = 0 \tag{3.22}$$

eq.[(3.21)] and eq.[(3.22)] are the Virasoro constraints. For open string with Neumann condition $\partial_\sigma X^\mu(\sigma = 0, \pi; \tau) = 0$. Eq.[(3.20)] can be immediately solved (x^μ, v^μ) can be arbitrary constants):

$$X^\mu(\sigma, \tau) = x^\mu + v^\mu \tau + X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \tag{3.23}$$

For closed strings X_R, X_L are independent periodic functions of 2π. For open strings applying Neumann condition, we get

$$X'_L(\tau) = X'_R(\tau) \text{ at } \sigma = 0 \tag{3.24}$$

$$X'_L(\tau - \pi) = X'_R(\tau + \pi) \text{ at } \sigma = \pi \tag{3.25}$$

We get $X_L = X_R$ and is periodic in 2π.

3.2.1 Light Cone Gauge

After fixing the worldsheet metric, one still have residual gauge freedom (conformal symmetry). Let's introduce:

$$\sigma^\pm = \frac{\tau \pm \sigma}{\sqrt{2}}; \quad ds^2 = -d\tau^2 + d\sigma^2 = -2d\sigma^+ d\sigma^- \tag{3.26}$$

hence this symmetry can be viewed as the preservation of $\gamma^{ab} = \eta^{ab}$ (up to a Weyl rescaling) as:

$$\sigma^+ \to \tilde{\sigma}^+ = f(\sigma^+); \quad \sigma^- \to \tilde{\sigma}^- = g(\sigma^-); \quad ds^2 \to -2\partial_+ f \partial_- g d\sigma^+ \sigma^- \tag{3.27}$$

$$\tilde{\tau} = \frac{f(\tau+\sigma) + g(\tau-\sigma)}{\sqrt{2}} \Rightarrow \tau \tag{3.28}$$

which has the same form as the classical solution of X^μ, then one can fix the gauge completely by choosing appropriate f and g so that:

$$\tau = \frac{X^+}{v^+}; \quad X^\pm = \frac{X^0 \pm X^1}{\sqrt{2}} \tag{3.29}$$

This is known as the light-cone gauge, as the worldsheet time is fixed by the spacetime light-cone coordinate. With $X^\mu = (X^+, X^-, X^i)$ (the transverse directions $i = 2, 3, \cdots, d-1$) : $dX^\mu dX_\mu = -2dX^+dX^- + dX^i dX^i$. In light-cone gauge, the Virasoro constraints [(3.21) and (3.22)] become:

$$2v^+\partial_\tau X^- = \left(\partial_\tau X^i\right)^2 + \left(\partial_\sigma X^i\right)^2 \tag{3.30}$$

$$v^+\partial_\sigma X^- = \partial_\sigma X^i \partial_\sigma X^i \tag{3.31}$$

The independence degrees of freedom are X^i. Since X^0 has a "wrong" sign for its kinetic terms, no X^0 in these degrees of freedom actually partly solve a problem of unitarity at quantum level.

Now eq.[(3.23)] expanding in Fourier series, for closed string we get

$$X^\mu(\sigma,\tau) = X^\mu + v^\mu\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}\right) \tag{3.32}$$

It's similar for open string, but from $X_R^\mu = X_L^\mu$ one arrives at $\alpha_n^\mu = \tilde{\alpha}_n^\mu$:

$$X^\mu(\sigma,\tau) = X^\mu + v^\mu\tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau} cosn\sigma \tag{3.33}$$

The center off mass motion can be found by averaging the position of the strings at a given time slice ($l = 2\pi$ for closed strings and $l = \pi$ for open strings):

$$\frac{1}{l}\int_0^l d\sigma X^\mu(\sigma,\tau) = x^\mu + v^\mu \tau \tag{3.34}$$

The constant v^μ is identified with the strings' center of mass velocity.

The classical coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ keep track of the oscillation modes of the strings. While the closed strings have independent left-moving and right-moving contributions, open string can be described as standing waves so that left-moving and right-moving are the same.

In the light-cone gauge, $X^+ = v^+\tau$ and X^- can be obtained by writing X^- in Fourier expansion, plugging equations [(3.32)] and [(3.33)] into equations [(3.30)] and [(3.31)] then equating the coefficients of different Fourier modes. The 0th (non-oscillating) mode gives the relations between the strings' center of mass velocity and the strings' oscillation modes. Now we get,

$$2v^+v^- = v_i^2 + 2\alpha' \sum_{m\neq 0} \alpha_{-m}^i \alpha_m^i \ \text{(open)} \tag{3.35}$$

$$2v^+v^- = v_i^2 + \alpha' \sum_{m\neq 0} \left(\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i\right) \ \text{(closed)} \tag{3.36}$$

Poincare global symmetries of the action corresponds to the conserved currents on the worldsheet. For the moment, let's look at translation and apply the standard Noether procedure:

$$\Pi_a^\mu = \frac{1}{2\pi\alpha'}\partial_a X^\mu \tag{3.37}$$

Also note that $\partial^a \Pi_a^\mu = 0$, from the equation of motion for X^μ. Π_τ^μ is the momentum density along the string, and the corresponded conserved current is the string momentum in space-time:

$$p^\mu = \int_0^l d\sigma \Pi^\mu_\tau = \frac{l}{2\pi}\frac{v^\mu}{\alpha'} \tag{3.38}$$

The mass-squared is related to the spacetime momentum of the strings (the mass shell condition): $M^2 = -p^\mu p_\mu = 2p^+p^- + p_i^2$

$$M^2 = \frac{1}{2\alpha'}\sum_{m\neq 0} \alpha^i_{-m}\alpha^i_m \ \ \text{(open)} \tag{3.39}$$

$$M^2 = \frac{1}{\alpha'}\sum_{m\neq 0} \left[\alpha^i_{-m}\alpha^i_m + \tilde{\alpha}^i_{-m}\tilde{\alpha}^i_m\right] \ \ \text{(closed)} \tag{3.40}$$

3.2.2 Quantization :

After understanding the strings at classical level, the next step is to quantization – quantize independent degrees of freedom $X^i(\sigma,\tau)$ (with canonical momentum density Π^i) in the action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial^a X^i \partial_a X^i; \quad \Pi^i = \frac{1}{2\pi\alpha'}\partial_\tau X^i \tag{3.41}$$

Nominate X^i to be a quantum operator, with the canonical commutation relation at a given time slice:

$$\left[X^i(\sigma,\tau), X^j(\sigma',\tau)\right] = \left[\Pi^i(\sigma,\tau), \Pi^j(\sigma',\tau)\right] = 0; \quad \left[X^i(\sigma,\tau), \Pi^j(\sigma',\tau)\right] = i\delta^{ij}\delta(\sigma-\sigma') \tag{3.42}$$

The results are 0th mode x^i, p^i and oscillation modes $\alpha^i, \tilde{\alpha}^i$ all become operators:

$$\left[x^i, p^j\right] = i\delta^{ij}; \quad \left[\alpha^i_n, \alpha^j_m\right] = \left[\tilde{\alpha}^i_n, \tilde{\alpha}^j_m\right] = n\delta^{ij}\delta_{n+m,0} \tag{3.43}$$

Note that $\alpha^i, \tilde{\alpha}^i$ can be related to the creation and annihilation operators :

$$\frac{1}{\sqrt{n}}\alpha^i_n = a^i_n; \quad \frac{1}{\sqrt{n}}\alpha^i_{-n} = \left(a^i_{-n}\right)^\dagger; \quad \frac{1}{\sqrt{n}}\tilde{\alpha}^i_{-n} = \left(\tilde{a}^i_{-n}\right)^\dagger; \quad \frac{1}{\sqrt{n}}\tilde{\alpha}^i_n = \tilde{a}^i_n \tag{3.44}$$

Therefore, the oscillator vacuum state (labelled by string space time momentum p^μ) satisfies :

$$\alpha^i_n \left|0, p^\mu\right\rangle = \tilde{\alpha}^i_n \left|0, p^\mu\right\rangle = 0, \quad n > 0 \tag{3.45}$$

Excited states can be built from creation operators ($\alpha^i_{-n}, \tilde{\alpha}^i_{-n}$ with $n > 0$) :

$$\alpha^{i_1}_{-n_1}\alpha^{i_2}_{-n_2}\cdots\tilde{\alpha}^{j_1}_{-m_1}\tilde{\alpha}^{j_2}_{-m_2}\cdots\left|0, p^\mu\right\rangle \tag{3.46}$$

For closed string, define the oscillation number operator (no summation in i index, and the order of operators is very important):

$$N^i_n = \frac{1}{n}\alpha^i_{-n}\alpha^i_n; \quad \tilde{N}^i_n = \frac{1}{n}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \tag{3.47}$$

The quantum version of the mass shell condition for closed strings,

$$M^2 = \frac{2}{\alpha'}\sum_{i=2}^{D-1}\sum_{n\neq 0} n\left(N^i_n + \tilde{N}^i_n\right) + a_0 \tag{3.48}$$

The constant a_0 is the zero-point energy for closed string, comes from rearranging the operator to normal ordered :

$$a_0 = \frac{2(D-2)}{\alpha'} \sum_{n=1}^{\infty} n = -\frac{(D-2)}{24} \frac{4}{\alpha'} \tag{3.49}$$

Similarly for open string,

$$M^2 = \frac{1}{\alpha'} \sum_{i=2}^{D-1} \sum_{m=1}^{\infty} m N_m^i + a_0 \tag{3.50}$$

The constant a_0 is found to be :

$$a_0 = \frac{(D-2)}{2\alpha'} \sum_{n=1}^{\infty} n = -\frac{(D-2)}{24} \frac{1}{\alpha'} \tag{3.51}$$

The string-spectrum can be read-off, as each state of a string corresponds to a spactime particle state.

Let's start with the particles content of open strings:

1. The oscillation vacuum state is:

$$|0, p^\mu\rangle\,; \;\; N_m^i = 0; \;\; \forall m, i \tag{3.52}$$

   The spacetime transformation of this state indicates that it should be a spacetime scalar of mass: $M^2 = -\frac{D-2}{24\alpha'}$. This particle is a tachyon when $D > 2$ as $M^2 < 0$.

2. The oscillation 1st excited state transforms as a $SO(D-2)$ vector under spacetime rotation: $\alpha_{-1}^i |0, p^\mu\rangle$, $M^2 = -\frac{D-2}{24\alpha'}$. Since the 1st excited state

is a vector with only $D-2$ independent component, for the consistency of Lorentz symmetries or the quantization procedure, then it should be massless: $M^2 = 0 \Rightarrow D = 26$. This results for spacetime dimension is known as the critical dimension of (bosonic) string theory. The coherent state created by many of this particle in spacetime gives the field configuration of a massless vector field A_μ, later play the roles of a gauge boson (like photon and gluon).

3. Higher excitations are all massive giving by multiplets of mass-squared spacing $\frac{1}{\alpha'}$. For example, the particle description for the 2nd excited string state (massive): $\alpha^i_{-1}\alpha^j_{-1}|0,p^\mu\rangle$, $\alpha^i_{-2}|0,p^\mu\rangle$, $M^2 = \frac{1}{\alpha'}$

The particles content of closed strings can also be read-off similarly:

1. The oscillation vacuum state is:

$$|0,p^\mu\rangle\,; \;\; N^i_m = \tilde{N}^i_m = 0; \;\; \forall m, i \tag{3.53}$$

The spacetime transformation of this state indicates that it should be a spacetime scalar of mass: $M^2 = -\frac{D-2}{6\alpha'}$. Similar to open string, this particle is a tachyon when $D > 2$ as $M^2 < 0$.

2. The oscillation 1st excited state is (using the level matching condition): $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0,p^\mu\rangle$, $M^2 = \frac{26-D}{6\alpha'}$. Again, only for $D = 26$, this state fall into the irreducible representations $SO(D-2)$ of Lorentz group consistently (as massless $M^2 = 0$ spin 2 particles) with further decomposition to scalar (trace), symmetric trace-less and anti-symmetric: $\sum_i^{24}\alpha^i_{-1}\tilde{\alpha}^i_{-1}|0,p^\mu\rangle$, $\alpha^i_{-1}\tilde{\alpha}^j_{-1}e_{ij}|0,p^\mu\rangle$, $\alpha^i_{-1}\tilde{\alpha}^j_{-1}b_{ij}|0,p^\mu\rangle$. The coherent state given from many of these particles in spacetime are the field

configuration of a massless scalar field Φ (dilaton), a traceless symmetric tensor field $G_{\mu\nu}$ (graviton) and a anti-symmetric tensor field $B_{\mu\nu}$ (Kalb-Ramon, or B-field), associated with gauge symmetries.

3. Higher excitations are all massive giving by multiplets of mass-squared spacing $\frac{4}{\alpha'}$. For example, the particle description for the 2nd excited string state (massive): $\alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-1}\tilde{\alpha}^l_{-1}\left|0,p^\mu\right\rangle$, $\alpha^i_{-2}\tilde{\alpha}^k_{-1}\tilde{\alpha}^l_{-1}\left|0,p^\mu\right\rangle$, $\alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-2}\left|0,p^\mu\right\rangle$, $M^2 = \frac{4}{\alpha'}$. This fits with the description of a massive spin 4 particles in $D = 26$ Lorentz irreducible group, form a complete multiplets of $SO(25)$. The same consistency holds for higher spin (mass) particles.

At low energy$E \ll \frac{1}{\alpha'}$, the dynamics of A_μ and $G_{\mu\nu}$ should be governed by Maxwell and Einstein theory. This is confirmed by explicit calculations of scattering amplitudes of these particles in string theory ($g_s \ll 1$). Also we get $G_N \propto g_s^2$ and $g_s = e^{\langle\Phi\rangle}$.

There are different quantizations exist (with no tachyon), and IIA and IIB are the name of perturbative superstring theories of future interests. Both of them require the critical dimension to be $The massless closed superstring (bosonic) fields in spacetime are :$.

The massless closed superstring (bosonic) fields in spacetime are:

1. For IIA superstring:

$$\Phi,\ G_{\mu\nu},\ B_{\mu\nu},\ A_\mu,\ C^{(3)}_{\mu\nu\lambda} \tag{3.54}$$

   A_μ and $C^{(3)}_{\mu\nu\lambda}$ are the RR fields.

2. For IIB superstring:

$$\Phi,\ G_{\mu\nu},\ B_{\mu\nu},\ \chi,\ C^{(2)}_{\mu\nu},\ C^{(4)}_{\mu\nu\lambda\sigma} \tag{3.55}$$

$\chi, C^{(2)}_{\mu\nu}$ and $C^{(4)}_{\mu\nu\lambda\sigma}$ are the RR fields.

The low energy effective field theories of these superstring theories are IIA and IIB supergravity.

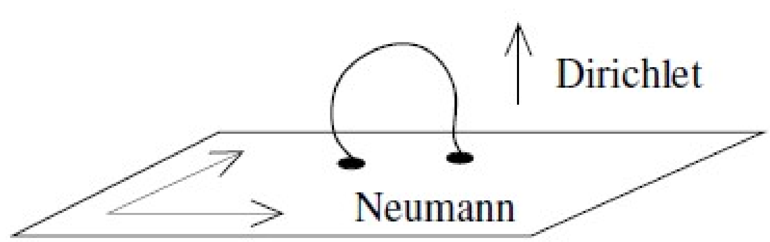
3.3 D-Branes :

We have two imporatant Boundary condiotions :

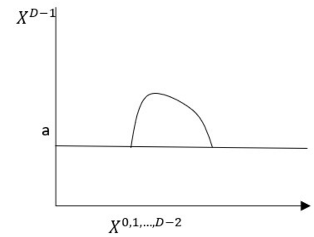
1. The Dirichlet condition (D-condition) for the ends of open string ($\sigma_{end} = 0, \pi$) in spatial direction, $\delta X^i = 0 \rightarrow X^i(\sigma_{end}, \tau) = \text{const}$.
2. The Neumann condition (N-condition) for the ends of open string ($\sigma_{end} = 0, \pi$), is $\partial_\sigma \delta X^\mu = 0$

Let's start with the D-condition,

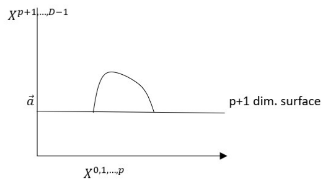
The physical interpretation is that the end point is restricted at a hypersurface or a p-dimensional surface, a "spacetime defect" where open strings can end, which seemingly is not a degrees of freedom from perturbative string point of view. Such object is called a D-brane, and D_p-brane is a D-brane with p spatial direction,



1. Suppose in all directions we have



$$X^{0,1,\cdots,D-2} = N \text{ for } \sigma = 0, \pi \quad (3.56)$$
$$X^{D-1} = a \text{ for } \sigma = 0, \pi \quad (3.57)$$

2. 

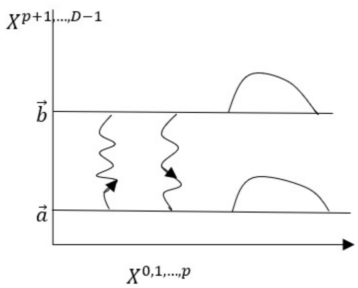
$$X^{0,1,\cdots,p} = N \text{ for } \sigma = 0, \pi \quad (3.58)$$
$$X^{p+1,p+2,\cdots,D-1} = \vec{a} \text{ for } \sigma = 0, \pi \quad (3.59)$$

3. If in all direction the end of open string has N-condition, one can interpret that this means the open string can end anywhere, which can be thought as there's a space-filling brane (D25-brane, as the number of spatial dimensions is 25 in bosonic string theory) and there's interesting dynamics associated with that nonperturbative objects. One way to think about it is that strings must naturally exist as closed strings, and they can only break open at some special places in spacetime, where D-branes are located.

4. In Lorentzian spacetime, however, there's no D(-1)-brane (in the sense that it's not a stable object, only appears for an instant in time). Because time in the target space cannot standstill, so the condition $X^0 =$ const. doesn't happen.

5. A D0-brane is a particle; a D1-brane is itself a string; a D2-brane a membrane and so on.

6. Consider more than one D-branes. Here we get 4-types of open strings.

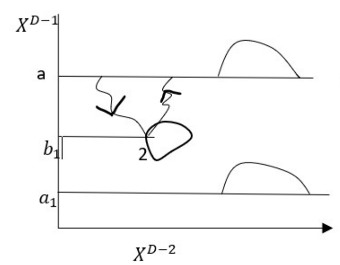


$$X^{0,1,\cdots,p} = N \text{ for } \sigma = 0, \pi \quad (3.60)$$

$$X^{p+1,p+2,\cdots,D-1} = \vec{a} \text{ for } (\sigma = 0, \tau) \quad (3.61)$$

$$= \vec{b} \text{ for } (\sigma = \pi, \tau) \quad (3.62)$$

7. D-branes of different dimensions



$$X^{D-2} = N \text{ for } (\sigma = 0, \tau) \quad (3.63)$$

$$X^{D-2} = a_1 \text{ for } (\sigma = \pi, \tau) \quad (3.64)$$

A $D_p - brane$ breaks translational and Lorentz symmetries of original $Mink_D$ to Poincare$(1, p) \times SO(D - 1 - p)$

What D-brane tells us ?

The D-brane taught us two things. First, the GKP-Witten relation becomes more precise:

$$Z_{\mathcal{N}_4} = Z_{AdS_5 \times S^5} \quad (3.65)$$

The left-hand side is the partition function of the $\mathcal{N} = 4$ SYM, and the right-hand side is the partition function of string theory on $AdS_5 \times S^5$.

Second, from the D-brane, we are able to obtain the AdS/CFT dictionary. But one often compactifies S^5 and consider the resulting five-dimensional gravitational theory. The theory obtained in this way is called gauged supergravity.

The actual procedure of the S^5 compactification is rather complicated, and the full gauged supergravity action is complicated as well.

Part II

Chapter 4

The AdS spacetime

So what is Anti-deSitter (AdS) spacetime?

AdS_{d+1} is a maximally symmetric spacetime with negative curvature. It is a solution to Einstein's equations with a negative cosmological constant.

4.1 Spacetimes with constant curvature

Now consider spacetimes with constant curvature. The AdS_2 spacetime can be embedded into a flat spacetime with two timelike directions as in figure[(4.1)]

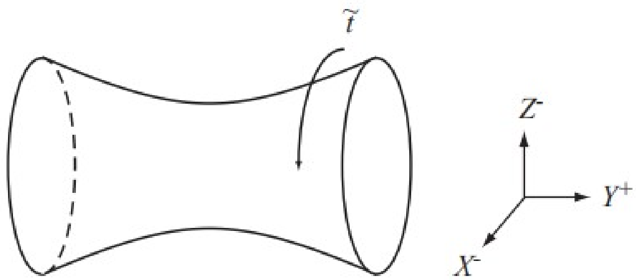


Figure 4.1: The embedding of AdS_2 into $\mathbb{R}^{2,1}$. The timelike direction $\tilde{t}$ is periodic, so we consider the covering space

$$ds^2 = -dZ^2 - dX^2 + dY^2 \tag{4.1}$$

$$-Z^2 - X^2 + Y^2 = L^2 \tag{4.2}$$

The parameter L is called the AdS radius. The AdS_2 spacetime has the $SO(2,1)$ invariance. Just like S^2 and H^2, take a coordinate system

$$Z = L\cosh\rho\cos\tilde{t}, \quad X = L\cosh\rho\sin\tilde{t}, \quad Y = L\sinh\rho \tag{4.3}$$

Then, the metric becomes

$$ds^2 = L^2(-\cosh^2\rho d\tilde{t}^2 + d\rho^2) \tag{4.4}$$

This coordinate system $(\tilde{t}, \rho)$ is called the global coordinates. Although we embed the AdS spacetime into a flat spacetime with two timelike directions X and Y, the AdS spacetime itself has only one timelike direction.

From Eq. (4.3), the coordinate $\tilde{t}$ has the periodicity 2π, so the timelike direction is periodic. This is problematic causally, so one usually unwraps the timelike dirrection and considers the covering space of the AdS_2 spacetime, where $-\infty < \tilde{t} < \infty$. The AdS spacetime in AdS/CFT is this covering space. The AdS_2 spacetime has a constant negative curvature $R = -2/L^2$.

One often considers the AdS_5 spacetime for applications to AdS/CFT, but for the dS spacetime, one often considers the dS_4 spacetime for applications to cosmology.

4.2 Various coordinate systems of AdS spacetime

So far we discussed the AdS spacetime using the global coordinates. But various other coordinate systems appear in the literature.

Static coordinates : $(\tilde{t}, \tilde{r})$ The coordinate $\tilde{r}$ is defined by $\tilde{r} = \sinh \rho$. The metric becomes

$$\frac{dS^2}{L^2} = -\left(\tilde{r}^2 + 1\right) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 + 1} \tag{4.5}$$

This coordinate system is useful to compare with the AdS black hole.

Conformal coordinates : $(\tilde{t}, \theta)$ the coordinate θ is defined by $\tan \theta = \sinh \rho (\theta : -\pi/2 \to \pi/2)$. The metric becomes flat up to an overfall factor (conformally flat) :

$$\frac{dS^2}{L^2} = \frac{1}{\cos^2 \theta} \left(-d\tilde{t}^2 + d\theta^2\right) \tag{4.6}$$

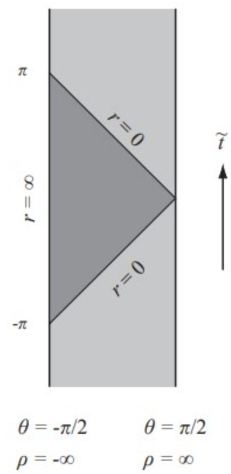


Figure 4.2: The AdS_2 spacetime in conformal coordinates. The Poincare coordinates cover only part of the full AdS spacetime which is shown in the dark shaded region (Poincare patch).

The AdS spacetime is represented as Fig. [(4.2)] in this coordinate system. What is important is the existence of the spatial "boundary" at $\theta = \pm\pi/2$. This boundary is called the AdS boundary.

Poincare coordinates : (t,r) this coordinate system is defined by

$$Z = \frac{Lr}{2}\left(-t^2 + \frac{1}{r^2} + 1\right) \tag{4.7}$$

$$X = Lrt \tag{4.8}$$

$$Y = \frac{Lr}{2}\left(-t^2 + \frac{1}{r^2} - 1\right) \tag{4.9}$$

$(r > 0, t : -\infty \to \infty)$. The metric becomes

$$\frac{dS^2}{L^2} = -r^2 dt^2 + \frac{dr^2}{r^2} \tag{4.10}$$

This is the most often used coordinate system in AdS/CFT. This coordinate system is also useful to compare with the AdS black hole.

4.3 Maximally symmetric spacetimes :

We saw that these spacetimes have a large number of symmetries like S^2. In fact, they are called maximally symmetric spacetimes which admit the maximum number of symmetry generators. As a familiar example, the Minkowski spacetime is also a spacetime with constant curvature (namely R = 0) and is a maximally symmetric space. The (p + 2)-dimensional Minkowski spacetime has the $ISO(1, p+1)$ Poincare invariance. The number of symmetry generators is (p + 1)(p + 2)/2 for $SO(1, p+1)$ and (p + 2) for translations, so (p + 2)(p + 3)/2 in total. This is the maximum number of generators.

Chapter 5

The AdS black hole

Black holes can exist in the AdS spacetime. The simplest AdS black hole is known as the Schwarzschild-AdS black hole (SAdS black hole hereafter). Just like the Schwarzschild black hole, one can consider AdS black holes with spherical horizon, but we consider AdS black holes with planar horizon or AdS black branes for the time being.

The $SAdS_5$ black hole is a solution of the Einstein equation with a negative cosmological constant like the AdS_5 spacetime. The metric is given by

$$ds_5^2 = -\left(\frac{r}{L}\right)^2 h(r)dt^2 + \frac{dr^2}{\left(\frac{r}{L}\right)^2 h(r)} + \left(\frac{r}{L}\right)^2 \left(dx^2 + dy^2 + dz^2\right) \tag{5.1}$$

$$h(r) = 1 - \left(\frac{r_0}{r}\right)^4 \tag{5.2}$$

The horizon is located at $r = r_0$. When $r_0 = 0$, the metric reduces to the AdS_5 spacetime in Poincare coordinates. The g_{00} component contains the factor $r_0^4/(L^2r^2)$.

The $\mathcal{O}(r^{-2})$ behavior comes from the Newtonian potential which behaves as r^{-2} in the five-dimensional spacetime. The coordinates (x, y, z) represent $\mathbb{R}^3$ coordinates. In the Schwarzschild black hole, this part was $r^2 d\Omega^2$ which represents a spherical horizon, but here the $r = r_0$ horizon extends indefinitely in (x, y, z)-directions. The AdS spacetime is a spacetime with constant curvature, but the $SAdS_5$ black hole is not. For example, there is a curvature singularity at r = 0.

5.1 Thermodynamic quantities of AdS black hole

Here, we compute thermodynamic quantities of the $SAdS_5$ black hole. In AdS/CFT, they are interpreted as thermodynamic quantities of the dual $\mathcal{N} = 4$ SYM at strong coupling. In order to rewrite black hole results as gauge theory results, one needs the relation of the parameters between two theories.

$$N_c^2 = \frac{\pi}{2}\frac{L^3}{G_5}, \quad \lambda = \left(\frac{L}{l_s}\right)^4 \tag{5.3}$$

Such relations are known as the AdS/CFT dictionary. On the left-hand side, we have gauge theory parameters which are written in terms of gravity parameters on the right-hand side.

First, the temperature is given by

$$\begin{aligned} T &= \frac{f'(r_0)}{4\pi} && (5.4)\\ &= \frac{1}{4\pi}\frac{1}{L^2}\left(2r + \frac{2r_0^4}{r^3}\right)\Big|_{r=r_0} && (5.5)\\ &= \frac{1}{\pi}\frac{r_0}{L^2} && (5.6)\end{aligned}$$

For this black hole, the horizon has an infinite extension, and the entropy itself diverges, so it is more appropriate to use the entropy density s. Let the spatial extension of the black hole as $0 \leq x, y, z \leq L_x, L_y, L_z$. (This is just an infrared cutoff to avoid divergent expressions.) The gauge theory coordinates are (x, y, z), so the gauge theory volume is $V_3 := L_x L_y L_z$. This is different from the horizon "area" since the line element is $(r/L)^2(dx^2 + dy^2 + dz^2)$. Then, from the area law

$$S = \frac{A}{4G_5} = \frac{1}{4G_5}\left(\frac{r_0}{L}\right)^3 V_3 \tag{5.7}$$

$$\Rightarrow s = \frac{S}{V_3} = \frac{1}{4G_5}\left(\frac{r_0}{L}\right)^3 \tag{5.8}$$

$$= \frac{a}{4G_5} \tag{5.9}$$

where $a = A/V_3$ is the "horizon area density". Using the temperature (5.6) and the AdS/CFT dictionary, one gets

$$\boxed{s = \frac{\pi^2}{2} N_c^2 T^3 \qquad (5.10)}$$

The rest of thermodynamic quantities can be determined using thermodynamic relations. The first law $d\varepsilon = T ds$ can determine the energy density

$$\varepsilon = \frac{3}{8}\pi^2 N_c^4 T^3 \tag{5.11}$$

The Euler relation $\varepsilon = Ts - P$ then determines the pressure $P = \frac{1}{3}\varepsilon$.

5.2 AdS Black Holes and Thermality

Let us begin with a thought experiment. Consider a CFT living on the Lorentzian cylinder $R \times S^{d-1}$, and let us slowly heat it up. Since the dilatation operator serves as the Hamiltonian, as we increase the temperature the CFT will be in a state characterized by larger and larger operator/state dimensions.

Since the AdS Hilbert space is identical to that of the CFT, we can interpret our hot CFT as a thermal state in AdS. But what will this state consist of? At low temperatures we will just have a thermal gas made up of the light particles in AdS. Due to the AdS geometry, these particles will mostly move around near the center of AdS, with only occasional excursions further away. This means that as we increase the temperature, we will be cramming more and more energy into a region of roughly fixed size. In the presence of dynamical gravity, this cannot go on forever – eventually, at some critical temperature T_c, the hot gas will collapse to form a black hole in AdS. Our thought experiment shows that black holes in AdS must correspond to a hot CFT!

Chapter 6

Properties of AdS Black Holes

6.1 How anti-de Sitter black holes reach thermal equilibrium

Black holes in anti-de Sitter (AdS) spacetimes have been well studied in recent decades due to their applications in holography (gauge/gravity duality). These black holes behave rather differently from their asymptotically flat counterpart. Notably, their event horizon need not be spherical, topologically speaking. Instead, black holes with hyperbolic or toroidal horizon are also valid solutions to the Einstein field equations. Regardless of their horizon topology, AdS black holes possess very different thermodynamical behavior compared to the asymptotically flat ones. The Hawking temperature in d-dimensions takes the form (in the units $G = c = \hbar = k_B = 1$)

$$T = \frac{k(d-3)L^2 + (d-1)r_h^2}{4\pi L^2 r_h}, \tag{6.1}$$

where $k = +1, 0, -1$ correspond to horizons that are positively curved, flat, and negatively curved, respectively, and r_h denotes the radial location of the event horizon. For a sufficiently large black hole, namely those with horizon size larger than the AdS curvature length scale ($r_h > L$), the temperature is directly proportional to r_h. That is to say, large AdS black holes are "hot"[1]. This lies in stark contrast with the asymptotically flat Schwarzschild black hole, whose temperature scales inversely proportional to its mass (and therefore size).

In addition, asymptotically locally AdS spacetimes have a timelike boundary at spatial infinity. Remarkably, null geodesics from within the bulk can hit the boundary and be reflected back in a finite affine parameter interval (and also in a finite coordinate time t, if we use the canonical Schwarzschild-like coordinates). To see this, let us focus on the $k = 0$ case, which is widely used in holography. Hereinafter, we shall refer to such black holes as "flat black holes".

Suppose there is no black hole. The metric tensor

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + r^2\left(\sum_{i=1}^{d-2}(dx^i)^2\right), \tag{6.2}$$

simply describes a flat foliation of the maximally symmetric AdS spacetime. This coordinate system fails at the center $r = 0$, so let us consider $r = \varepsilon > 0$, where ε is small. The proper time between any two events both located at $r = \varepsilon$ is $(\varepsilon/L)\Delta t$,

[1] Even though the temperature of AdS black holes can be arbitrarily high from the viewpoint of the global geometry, local observers never see thermal radiation at such Hawking temperature. Keeping this subtlety in mind, we shall no longer put scare quotes around the words hot or cold hereinafter.

where Δt for a photon that goes from $r = \varepsilon$ to ∞ and back is

$$\Delta t = 2 \int_{\varepsilon}^{\infty} \frac{L^2}{r^2} \mathrm{r} = \frac{2L^2}{\varepsilon}. \tag{6.3}$$

This is finite, although large. The proper time elapsed for the static observer is $2L$. Note that ε drops out in the proper time, as it should, since the AdS "center" is arbitrary. As a consequence, if a reflective boundary condition is imposed, the Hawking photons will be reflected back into the black hole and so a sufficiently large black hole can attain thermal equilibrium. (Another consequence is the non-linear instability of AdS: surprisingly a large class of arbitrarily small perturbations can be reflected and refocused in the bulk, thus causing black hole formation)

For the $k = 1$ case, small black holes are also hot ($T \sim 1/r_h$ as can be seen from Eq.(6.1)), much like a small asymptotically flat Schwarzschild black hole. Since Hawking radiation takes time to hit the boundary and be reflected back, such small black holes can therefore completely evaporate before they have any hope to achieve thermal equilibrium. In other words, large black holes (which have positive specific heat) are stable while small black holes (which have negative specific heat) are therefore unstable. We could in principle use this stability criterion to define "large" and "small". While this criterion happens to coincide with using either the mass or the horizon size being greater than L to define the black hole "size" in the $k = 1$ case, it does not hold for the $k = 0$ case that we would like to focus on in this work. These black holes have Hawking temperature that is proportional to r_h regardless of the black hole size, this means that small flat black holes are cold, i.e., their rate of evaporation is slow. Therefore it is not impossible for small black holes to attain thermal equilibrium with their Hawking radiation. *All* $k = 0$ black holes would therefore be "large" if we were to use the stability/specific heat to define its "size". This is why we use $r_h > L$ as the definition of a large black hole.

Furthermore, since the boundary condition can be changed to a completely absorptive one in which there is no thermal equilibrium, we prefer to use a definition that holds independent of the boundary condition. That is, a large black hole would remain large even if we change the boundary condition. Given a fixed black hole in the bulk, it takes time for the radiation to reach the boundary and come back. Until the radiation reaches the boundary (and potentially reflected back or absorbed depending on the boundary condition), the black hole has no knowledge of whether it can reach equilibrium. So a local criteria that allows us to define the black hole size at any given time, even *before* the first Hawking radiation is emitted (so that we can speak of whether an *initially* "large" black hole can evolve into a "small" one or remains "large", even in the $k = 1$ case) is more useful. As we shall see, defined this way, large flat black holes can evaporate into a small black hole which is in thermal equilibrium with their Hawking quanta[2].

6.2 Perturbations of anti-de Sitter black holes

6.2.1 Introduction

The perturbations of a black hole are governed by quasi-normal modes (QNMs). The latter are typically obtained by solving a wave equation for small fluctuations

[2]Gibbons and Perry argued that black holes can remain in thermal equilibrium with a heat bath even in the presence of particle interactions, though his work is restricted to the asymptotically flat case, the conclusion is likely to be generic

in the black hole background subject to the conditions that the flux be ingoing at the horizon and outgoing at asymptotic infinity.

In this section I discuss scalar, gravitational and electromagnetic perturbations of an AdS Schwarzschild black hole in d dimensions analytically calculating the QNM spectrum in the high frequency regime. Low overtones will be discussed in the next section. The metric of an AdS Schwarzschild black hole is

$$ds^2 = -\left(\frac{r^2}{l^2} + K - \frac{2\mu}{r^{d-3}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2} + K - \frac{2\mu}{r^{d-3}}} + r^2 d\Sigma^2_{K,d-2} \tag{6.4}$$

I shall choose units so that the AdS radius l = 1. The horizon radius and Hawking temperature are, respectively,

$$2\mu = r_+^{d-1}\left(1 + \frac{K}{r_+^2}\right), \;\; T_H = \frac{(d-1)r_+^2 + K(d-3)}{4\pi r+} \tag{6.5}$$

The mass and entropy of the hole are, respectively,

$$M = (d-2)(K + r_+^2)\frac{r_+^{d-3}}{16\pi G} Vol(\Sigma_{K,d-2}), \;\; S = \frac{r_+^{d-2}}{4G} Vol(\Sigma_{K,d-2}) \tag{6.6}$$

The parameter K determines the curvature of the horizon and the boundary of AdS space. For $K = 0, +1, -1$ we have, respectively, a flat ($\mathbb{R}^{d-2}$), spherical ($\mathbb{S}^{d-2}$) and hyperbolic ($\mathbb{H}^{d-2}/\Gamma$, topological black hole, where Γ is a discrete group of isometries) horizon (boundary).

The harmonics on $\Sigma_{K,d-2}$ satisfy

$$(\nabla^2 + k^2)\mathbb{T} = 0 \tag{6.7}$$

For K = 0, k is the momentum; for K = +1, the eigenvalues are quantized,

$$k^2 = l(l + d - 3) - \delta \tag{6.8}$$

whereas for K = -1,

$$k^2 = \xi^2 + \left(\frac{d-3}{2}\right)^2 + \delta \tag{6.9}$$

where ξ is discrete for non-trivial Γ. d = 0,1,2 for scalar, vector, or tensor perturbations, respectively.

6.2.2 Scalar perturbations

To find the asymptotic form of QNMs, we need to find an approximation to the wave equation valid in the high frequency regime. In three dimensions the resulting wave equation will be an exact equation (hypergeometric equation). In five dimensions, I shall turn the Heun equation into a hypergeometric equation which will lead to an analytic expression for the asymptotic form of QNM frequencies in agreement with numerical results.

AdS_5 **:**

Restricting attention to the case of a large black hole, the massless scalar wave equation reads

$$\frac{1}{r^3}\partial_r(r^5 f(r)\partial_r \Phi) - \frac{1}{r^2 f(r)}\partial_t^2\Phi - \frac{1}{r^2}\vec{\nabla}^2\Phi = 0, \quad f(r) = 1 - \frac{r_+^4}{r^4} \tag{6.10}$$

Writing the solution in the form

$$\Phi = e^{i(\omega t - \vec{p}\cdot\vec{x})}\Psi(y), \quad y = \frac{r^2}{r_+^2} \tag{6.11}$$

the radial wave equation becomes

$$(y^2-1)(y(y^2-1)\Psi')' + \left(\frac{\omega^2}{4}y^2 - \frac{p^2}{4}(y^2-1)\right)\Psi = 0 \tag{6.12}$$

For QNMs, we are interested in the analytic solution which vanishes at the boundary and behaves as an ingoing wave at the horizon. The wave equation contains an additional (unphysical) singularity at y = -1, at which the wavefunction behaves as $\Psi \sim (y+1)^{\pm\omega/4}$. Isolating the behavior of the wavefunction near the singularities $y = \pm 1$,

$$\Psi(y) = (y-1)^{-i\omega/4}(y+1)^{\pm\omega/4}F_\pm(y) \tag{6.13}$$

$F_\pm(y)$ satisfies the Heun equation

$$y(y^2-1)F''_\pm + \left[\left(3 - \frac{i\pm 1}{2}\omega\right)y^2 - \frac{i\pm 1}{2}\omega y - 1\right]F'_\pm + \left[\frac{\omega}{2}\left(\pm\frac{i\omega}{4} \mp 1 - i\right)y - (i\mp 1)\frac{\omega}{4} - \frac{p^2}{4}\right]F_\pm = 0 \tag{6.14}$$

to be solved in a region in the complex y-plane containing $|y| \geq 1$ which includes the physical regime $r > r_+$. For large ω , the constant terms in the polynomial coefficients of F' and F are small compared with the other terms, therefore they may be dropped. The wave equation may then be approximated by a hypergeometric equation

$$(y^2-1)F''_\pm + \left[\left(3 - \frac{i\pm 1}{2}\omega\right)y - \frac{i\pm 1}{2}\omega\right]F'_\pm + \frac{\omega}{2}\left(\pm\frac{i\omega}{4} \mp 1 - i\right)F_\pm = 0 \tag{6.15}$$

in the asymptotic limit of large frequencies ω. The acceptable solution is

$$F_0(x) = F_1(a_+; a_-; c; (y+1)/2), \;\; a_\pm = 1 - \frac{i \pm 1}{4}\omega \pm 1, \;\; c = \frac{3}{2} \pm \frac{\omega}{2} \tag{6.16}$$

For proper behavior at the boundary ($y \to \infty$), we demand that F be a polynomial, which leads to the condition $a_+ = -n, \;\; n = 1, 2, \ldots$ Indeed, it implies that F is a polynomial of order n, so as $y \to \infty, F \sim y^n \sim y^{-a_+}$ and $\Psi \sim y^{-i\omega/4} y^{\pm\omega/4} y^{-a_+} \sim y^{-2}$, as expected.

We deduce the quasi-normal frequencies

$$\hat{\omega} = \frac{\omega}{4\pi T_H} = 2n(\pm 1 - i) \tag{6.17}$$

6.2.3 Gravitational perturbations

Next I consider gravitational perturbations. For definiteness, I concentrate on the case of spherical black holes (K = +1). I shall derive analytic expressions for QNMs including first-order corrections. The results are in good agreement with results of numerical analysis. Extension to other forms of the horizon is straightforward. The radial wave equation for gravitational perturbations in the black-hole background can be cast into a Schrodinger-like form,

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \tag{6.18}$$

in terms of the tortoise coordinate defined by $\frac{dr_*}{dr} = \frac{1}{f(r)}$ The potential V for the various types of perturbation has been found by Ishibashi and Kodama. For tensor, vector and scalar perturbations, one obtains, respectively

$$V_T(r) = f(r)\Big[\frac{l(l+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\Big] \tag{6.19}$$

$$V_v(r) = f(r)\Big[\frac{l(l+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{rf'''(r)}{2(d-3)}\Big] \tag{6.20}$$

$$\begin{aligned} V_s(r) = &\frac{f(r)}{4r^2}\Big[l(l+d-3)-(d-2)+\frac{(d-1)(d-2)\mu}{r^{d-3}}\Big]^{-2} \\ &\times\Big\{\frac{d(d-1)^2(d-2)^3\mu^2}{R^2r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[l(l+d-3)-(d-2)]\mu}{R^2r^{d-5}} \\ &+ \frac{(d-4)(d-6)[l(l+d-3)-(d-2)]^2r^2}{R^2} + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\ &+ \frac{4(d-1)(d-2)(2d^2-11d+18)[l(l+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\ &+ \frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} - \frac{6(d-2)(d-6)[l(l+d-3)-(d-2)^2]\mu}{r^{d-3}} \\ &- \frac{6(d-1)(d-2)^2(d-4)[l(l+d-3)-(d-2)]\mu}{r^{d-3}} \\ &+ 4[l(l+d-3)-(d-2)]^3 + d(d-2)[l(l+d-3)-(d-2)]^2\Big\} \end{aligned} \tag{6.21}$$

Near the black hole singularity ($r \sim 0$),

$$V_{\mathsf{T}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{T}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots\,, \quad \mathcal{A}_{\mathsf{T}} = \frac{(d-3)^2}{2(2d-5)} + \frac{\ell(\ell+d-3)}{d-2}, \tag{6.22}$$

$$V_{\mathsf{V}} = \frac{3}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{V}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots\ , \quad \mathcal{A}_{\mathsf{V}} = \frac{d^2-8d+13}{2(2d-15)} + \frac{\ell(\ell+d-3)}{d-2} \tag{6.23}$$

and

$$V_{\mathsf{S}} = -\frac{1}{4r_*^2} + \frac{\mathcal{A}_{\mathsf{S}}}{[-2(d-2)\mu]^{\frac{1}{d-2}}} r_*^{-\frac{d-1}{d-2}} + \dots\,, \tag{6.24}$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(2d^3 - 24d^2 + 94d - 116)}{4(2d-5)(d-2)} + \frac{(d^2 - 7d + 14)[\ell(\ell + d - 3) - (d-2)]}{(d-1)(d-2)^2} \tag{6.25}$$

I have included only the terms which contribute to the order I am interested in. The behavior of the potential near the origin may be summarized by

$$V = \frac{j^2 - 1}{4r_*^2} + \mathcal{A}\, r_*^{-\frac{d-1}{d-2}} + \dots \tag{6.26}$$

where $j = 0$ (2) for scalar and tensor (vector) perturbations.

On the other hand, near the boundary (large r),

$$V = \frac{j_\infty^2 - 1}{4(r_* - \bar{r}_*)^2} + \dots \quad , \quad \bar{r}_* = \int_0^\infty \frac{dr}{f(r)} \tag{6.27}$$

where $j_\infty = d - 1$, $d - 3$ and $d - 5$ for tensor, vector and scalar perturbations, respectively.

After rescaling the tortoise coordinate ($z = \omega r_*$), the wave equation to first order becomes

$$\left(\mathcal{H}_0 + \omega^{-\frac{d-3}{d-2}}\, \mathcal{H}_1\right) \Psi = 0, \tag{6.28}$$

where

$$\mathcal{H}_0 = \frac{d^2}{dz^2} - \left[\frac{j^2 - 1}{4z^2} - 1\right] \quad , \quad \mathcal{H}_1 = -\mathcal{A}\, z^{-\frac{d-1}{d-2}}. \tag{6.29}$$

By treating $\mathcal{H}_1$ as a perturbation, one may expand the wave function

$$\Psi(z) = \Psi_0(z) + \omega^{-\frac{d-3}{d-2}}\, \Psi_1(z) + \dots \tag{6.30}$$

and solve the wave equation perturbatively.

The zeroth-order wave equation,

$$\mathcal{H}_0 \Psi_0(z) = 0, \tag{6.31}$$

may be solved in terms of Bessel functions,

$$\Psi_0(z) = A_1 \sqrt{z}\, J_{\frac{j}{2}}(z) + A_2 \sqrt{z}\, N_{\frac{j}{2}}(z). \tag{6.32}$$

For large z, it behaves as

$$\begin{aligned} \Psi_0(z) &\sim \sqrt{\frac{2}{\pi}}\, [A_1 \cos(z - \alpha_+) + A_2 \sin(z - \alpha_+)] \\ &= \frac{1}{\sqrt{2\pi}} (A_1 - iA_2) e^{-i\alpha_+} e^{iz} + \frac{1}{\sqrt{2\pi}} (A_1 + iA_2) e^{+i\alpha_+} e^{-iz} \end{aligned}$$

where $\alpha_\pm = \frac{\pi}{4}(1 \pm j)$.

At the boundary ($r \to \infty$), the wavefunction ought to vanish, therefore the acceptable solution is

$$\Psi_0(r_*) = B \sqrt{\omega(r_* - \bar{r}_*)}\; J_{\frac{j_\infty}{2}}(\omega(r_* - \bar{r}_*)) \tag{6.33}$$

Indeed, $\Psi \to 0$ as $r_* \to \bar{r}_*$, as desired. Asymptotically (large z), it behaves as

$$\Psi(r_*) \sim \sqrt{\frac{2}{\pi}}\, B \cos\left[\omega(r_* - \bar{r}_*) + \beta\right] \ , \qquad \beta = \frac{\pi}{4}(1 + j_\infty) \tag{6.34}$$

This ought to be matched to the asymptotic form of the wavefunction in the vicinity of the black-hole singularity along the Stokes line $\Im z = \Im(\omega r_*) = 0$. This leads to a constraint on the coefficients $A_1,\ A_2$,

$$A_1 \tan(\omega \bar{r}_* - \beta - \alpha_+) - A_2 = 0. \tag{6.35}$$

By imposing the boundary condition at the horizon

$$\Psi(z) \sim e^{iz} \quad , \qquad z \to -\infty \ , \tag{6.36}$$

one obtains a second constraint. To find it, one needs to analytically continue the wavefunction near the black hole singularity ($z = 0$) to negative values of z. A rotation of z by $-\pi$ corresponds to a rotation by $-\frac{\pi}{d-2}$ near the origin in the complex r-plane. Using the known behavior of Bessel functions

$$J_\nu(e^{-i\pi} z) = e^{-i\pi\nu} J_\nu(z) \ , \qquad N_\nu(e^{-i\pi} z) = e^{i\pi\nu} N_\nu(z) - 2i \cos \pi\nu \, J_\nu(z) \tag{6.37}$$

for $z < 0$ the wavefunction changes to

$$\Psi_0(z) = e^{-i\pi(j+1)/2} \sqrt{-z} \ \left\{ \left[A_1 - i(1 + e^{i\pi j}) A_2 \right] \ J_{\frac{j}{2}}(-z) + A_2 e^{i\pi j} \ N_{\frac{j}{2}}(-z) \right\} \tag{6.38}$$

whose asymptotic behavior is given by

$$\Psi \sim \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} \left[A_1 - i(1 + 2e^{j\pi i}) A_2 \right] \ e^{-iz} + \frac{e^{-i\pi(j+1)/2}}{\sqrt{2\pi}} \left[A_1 - iA_2 \right] \ e^{iz} \tag{6.39}$$

Therefore one obtains a second constraint

$$A_1 - i(1 + 2e^{j\pi i}) A_2 = 0 \ \ . \tag{6.40}$$

The two constraints are compatible provided

$$\begin{vmatrix} 1 & -i(1+2e^{j\pi i}) \\ \tan(\omega\bar{r}_* - \beta - \alpha_+) & -1 \end{vmatrix} = 0 \tag{6.41}$$

which yields the quasi-normal frequencies

$$\omega\bar{r}_* = \frac{\pi}{4}(2+j+j_\infty) - \tan^{-1}\frac{i}{1+2e^{j\pi i}} + n\pi \tag{6.42}$$

The first-order correction to the above asymptotic expression may be found by standard perturbation theory. To first order, the wave equation becomes

$$\mathcal{H}_0\Psi_1 + \mathcal{H}_1\Psi_0 = 0 \tag{6.43}$$

The solution is

$$\Psi_1(z) = \sqrt{z}\,N_{\frac{j}{2}}(z)\int_0^z dz'\frac{\sqrt{z'}\,J_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} - \sqrt{z}\,J_{\frac{j}{2}}(z)\int_0^z dz'\frac{\sqrt{z'}\,N_{\frac{j}{2}}(z')\mathcal{H}_1\Psi_0(z')}{\mathcal{W}} \tag{6.44}$$

where $\mathcal{W} = 2/\pi$ is the Wronskian.

The wavefunction to first order reads

$$\Psi(z) = \{A_1[1-b(z)] - A_2a_2(z)\}\sqrt{z}J_{\frac{j}{2}}(z) + \{A_2[1+b(z)] + A_1a_1(z)\}\sqrt{z}N_{\frac{j}{2}}(z) \tag{6.45}$$

where

$$
\begin{aligned}
a_1(z) &= \frac{\pi\mathcal{A}}{2}\,\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\; z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') J_{\frac{j}{2}}(z') \\
a_2(z) &= \frac{\pi\mathcal{A}}{2}\,\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\; z'^{-\frac{1}{d-2}} N_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z') \\
b(z) &= \frac{\pi\mathcal{A}}{2}\,\omega^{-\frac{d-3}{d-2}} \int_0^z dz'\; z'^{-\frac{1}{d-2}} J_{\frac{j}{2}}(z') N_{\frac{j}{2}}(z')
\end{aligned}
$$

and $\mathcal{A}$ depends on the type of perturbation.

Asymptotically, it behaves as

$$\Psi(z) \sim \sqrt{\frac{2}{\pi}}\left[A_1' \cos(z-\alpha_+) + A_2' \sin(z-\alpha_+)\right] , \tag{6.46}$$

where

$$A_1' = [1-\bar{b}]A_1 - \bar{a}_2 A_2 \quad , \qquad A_2' = [1+\bar{b}]A_2 + \bar{a}_1 A_1 \tag{6.47}$$

and I introduced the notation

$$\bar{a}_1 = a_1(\infty) \quad , \qquad \bar{a}_2 = a_2(\infty) \quad , \qquad \bar{b} = b(\infty) \ . \tag{6.48}$$

The first constraint is modified to

$$A_1' \tan(\omega\bar{r}_* - \beta - \alpha_+) - A_2' = 0 \tag{6.49}$$

Explicitly,

$$[(1-\bar{b})\tan(\omega\bar{r}_* - \beta - \alpha_+) - \bar{a}_1]A_1 - [1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_* - \beta - \alpha_+)]A_2 = 0 \tag{6.50}$$

To find the second constraint to first order, one needs to approach the horizon. This entails a rotation by $-\pi$ in the z-plane. Using

$$
\begin{aligned}
a_1(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}a_1(z) \ , \\
a_2(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}\left[e^{i\pi j}a_2(z) - 4\cos^2\frac{\pi j}{2}a_1(z) - 2i(1+e^{i\pi j})b(z)\right] \ , \\
b(e^{-i\pi}z) &= e^{-i\pi\frac{d-3}{d-2}}\left[b(z) - i(1+e^{-i\pi j})a_1(z)\right]
\end{aligned}
$$

in the limit $z \to -\infty$ one obtains

$$\Psi(z) \sim -ie^{-ij\pi/2}B_1\cos(-z-\alpha_+) - ie^{ij\pi/2}B_2\sin(-z-\alpha_+) \tag{6.51}$$

where

$$
\begin{aligned}
B_1 = A_1 &- A_1e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] \\
&-A_2e^{-i\pi\frac{d-3}{d-2}}\left[e^{+i\pi j}\bar{a}_2 - 4\cos^2\frac{\pi j}{2}\bar{a}_1 - 2i(1+e^{+i\pi j})\bar{b}\right] \\
&-i(1+e^{i\pi j})\left[A_2 + A_2e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1\right] \\
B_2 = A_2 &+ A_2e^{-i\pi\frac{d-3}{d-2}}[\bar{b} - i(1+e^{-i\pi j})\bar{a}_1] + A_1e^{-i\pi\frac{d-3}{d-2}}e^{-i\pi j}\bar{a}_1
\end{aligned}
$$

Therefore the second constraint to first order reads

$$[1-e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1+\bar{b})]A_1-[i(1+2e^{i\pi j})+e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1+e^{i\pi j}\bar{a}_2-i\bar{b})]A_2 = 0 \tag{6.52}$$

Compatibility of the two first-order constraints yields

$$\begin{vmatrix} 1+\bar{b}+\bar{a}_2\tan(\omega\bar{r}_*-\beta-\alpha_+) & i(1+2e^{i\pi j})+e^{-i\pi\frac{d-3}{d-2}}((1+e^{i\pi j})\bar{a}_1+e^{i\pi j}\bar{a}_2-i\bar{b}) \\ (1-\bar{b})\tan(\omega\bar{r}_*-\beta-\alpha_+)-\bar{a}_1 & 1-e^{-i\pi\frac{d-3}{d-2}}(i\bar{a}_1+\bar{b}) \end{vmatrix}=0 \tag{6.53}$$

leading to the first-order expression for quasi-normal frequencies,

$$\begin{aligned}\omega\bar{r}_* = & \frac{\pi}{4}(2+j+j_\infty)+\frac{1}{2i}\ln 2+n\pi \\ & -\frac{1}{8}\left\{6i\bar{b}-2ie^{-i\pi\frac{d-3}{d-2}}\bar{b}-9\bar{a}_1+e^{-i\pi\frac{d-3}{d-2}}\bar{a}_1+\bar{a}_2-e^{-i\pi\frac{d-3}{d-2}}\bar{a}_2\right\}\end{aligned}$$

where

$$\begin{aligned}\bar{a}_1 &= \frac{\pi\mathcal{A}}{4}\left(\frac{n\pi}{2\bar{r}_*}\right)^{-\frac{d-3}{d-2}}\frac{\Gamma(\frac{1}{d-2})\Gamma(\frac{j}{2}+\frac{d-3}{2(d-2)})}{\Gamma^2(\frac{d-1}{2(d-2)})\Gamma(\frac{j}{2}+\frac{d-1}{2(d-2)})} \\ \bar{a}_2 &= \left[1+2\cot\frac{\pi(d-3)}{2(d-2)}\cot\frac{\pi}{2}\left(-j+\frac{d-3}{d-2}\right)\right]\bar{a}_1 \\ \bar{b} &= -\cot\frac{\pi(d-3)}{2(d-2)}\ \bar{a}_1\end{aligned}$$

Thus the first-order correction is $\sim\mathcal{O}(n^{-\frac{d-3}{d-2}})$.

The above analytic results are in good agreement with numerical results for a detailed comparison).

6.2.4 Electromagnetic perturbations

The electromagnetic potential in four dimensions is

$$V_{\mathsf{EM}}=\frac{\ell(\ell+1)}{r^2}f(r). \tag{6.54}$$

Near the origin,

$$V_{\mathsf{EM}} = \frac{j^2-1}{4r_*^2} + \frac{\ell(\ell+1)r_*^{-3/2}}{2\sqrt{-4\mu}} + \dots \ , \tag{6.55}$$

where $j = 1$. Therefore a vanishing potential to zeroth order is obtained. To calculate the QNM spectrum one needs to include first-order corrections from the outset. Working as with gravitational perturbations, one obtains the QNMs

$$\omega \bar{r}_* = n\pi - \frac{i}{4}\ln n + \frac{1}{2i}\ln\left(2(1+i)\mathcal{A}\sqrt{\bar{r}_*}\right) \ , \qquad \mathcal{A} = \frac{\ell(\ell+1)}{2\sqrt{-4\mu}} \tag{6.56}$$

Notice that the first-order correction behaves as $\ln n$, a fact which may be associated with gauge invariance.

As with gravitational perturbations, the above analytic results are in good agreement with numerical results for a detailed comparison).

6.2.5 Vector perturbations

I start with vector perturbations and work in the d-dimensional Schwarzschild background with $K = +1$ (spherical horizon and boundary). It is convenient to introduce the coordinate

$$u = \left(\frac{r_+}{r}\right)^{d-3} \tag{6.57}$$

The wave equation becomes

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u) \left(u^{\frac{d-4}{d-3}} \hat{f}(u) \Psi'\right)' + \hat{V}_{\mathsf{V}}(u)\Psi = \hat{\omega}^2 \Psi \ , \qquad \hat{\omega} = \frac{\omega}{r_+} \tag{6.58}$$

where prime denotes differentiation with respect to u and I have defined

$$\hat{f}(u) \equiv \frac{f(r)}{r^2} = 1 - u^{\frac{2}{d-3}}\left(u - \frac{1-u}{r_+^2}\right) \tag{6.59}$$

$$\hat{V}_{\mathsf{V}}(u) \equiv \frac{V_{\mathsf{V}}}{r_+^2} = \hat{f}(u)\left\{\hat{L}^2 + \frac{(d-2)(d-4)}{4}u^{-\frac{2}{d-3}}\hat{f}(u) - \frac{(d-1)(d-2)\left(1+\frac{1}{r_+^2}\right)}{2}u\right\} \tag{6.60}$$

where $\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_+^2}$.

First I consider the large black hole limit $r_+ \to \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small). Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} F(u) \tag{6.61}$$

the wave equation simplifies to

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega},\hat{L}}F = 0 \tag{6.62}$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4} u^{-\frac{2}{d-3}} \\
&\quad - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad - (d-3)\frac{i\hat{\omega}}{d-1} \frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

One may solve this equation perturbatively by separating

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0 \tag{6.63}$$

where

$$\begin{aligned}
\mathcal{H}_0 F &\equiv \mathcal{A}F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F \\
\mathcal{H}_1 F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F
\end{aligned}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \dots \tag{6.64}$$

at zeroth order the wave equation reads

$$\mathcal{H}_0 F_0 = 0 \tag{6.65}$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}} \tag{6.66}$$

being regular at both the horizon ($u = 1$) and the boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \to 0$ as $r \to \infty$). The Wronskian is

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})} \tag{6.67}$$

and another linearly independent solution is

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2} \tag{6.68}$$

which is unacceptable because it diverges at both the horizon ($\check{F}_0 \sim \ln(1-u)$ for $u \approx 1$) and the boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \to \infty$ as $r \to \infty$).

At first order the wave equation reads

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0 \tag{6.69}$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} \tag{6.70}$$

The limits of the inner integral may be adjusted at will because this amounts to adding an arbitrary amount of the unacceptable solution. To ensure regularity at the horizon, choose one of the limits of integration at $u = 1$ rendering the integrand regular at the horizon. Then at the boundary ($u = 0$),

$$F_1 = \check{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} + \text{regular terms} \tag{6.71}$$

The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} = 0 \tag{6.72}$$

which yields a constraint on the parameters (dispersion relation)

$$\mathbf{a}_0 \hat{L}^2 - i\mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0 \tag{6.73}$$

After some algebra, one arrives at

$$\mathbf{a}_0 = \frac{d-3}{d-1} \quad , \qquad \mathbf{a}_1 = d - 3 \tag{6.74}$$

The coefficient $\mathbf{a}_2$ may also be found explicitly for each dimension d, but it cannot be written as a function of d in closed form. It does not contribute to the dispersion relation at lowest order. E.g., for $d = 4, 5$, one obtains, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3}\ln 3 \quad , \qquad \frac{5}{6} - \frac{1}{2}\ln 2 \tag{6.75}$$

Eq. (6.73) is quadratic in $\hat{\omega}$ and has two solutions,

$$\hat{\omega}_0 \approx -i\frac{\hat{L}^2}{d-1} \quad , \qquad \hat{\omega}_1 \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\hat{L}^2}{d-1} \tag{6.76}$$

In terms of the frequency ω and the quantum number ℓ,

$$\omega_0 \approx -i\frac{\ell(\ell+d-3)}{(d-1)r_+} \quad , \quad \frac{\omega_1}{r_+} \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\ell(\ell+d-3)}{(d-1)r_+^2} \tag{6.77}$$

The smaller of the two, ω_0, is inversely proportional to the radius of the horizon and is not included in the asymptotic spectrum. The other solution, ω_1, is a crude estimate of the first overtone in the asymptotic spectrum, nevertheless it shares two important features with the asymptotic spectrum: it is proportional to r_+ and its dependence on ℓ is $\mathcal{O}(1/r_+^2)$. The approximation may be improved by including higher-order terms. This increases the degree of the polynomial in the dispersion relation (6.73) whose roots then yield approximate values of more QNMs. This method reproduces the asymptotic spectrum derived earlier albeit not in an efficient way.

To include finite size effects, I shall use perturbation theory (assuming $1/r_+$ is small) and replace $\mathcal{H}_1$ by

$$\mathcal{H}_1' = \mathcal{H}_1 + \frac{1}{r_+^2}\mathcal{H}_+ \tag{6.78}$$

where

$$\mathcal{H}_+ F \equiv \mathcal{A}_+ F'' + \mathcal{B}_+ F' + \mathcal{C}_+ F \tag{6.79}$$

The coefficients may be easily deduced by collecting $\mathcal{O}(1/r_+^2)$ terms in the exact wave equation. One obtains

$$\begin{aligned}
\mathcal{A}_+ &= -2(d-3)^2 u^2 (1-u) \\
\mathcal{B}_+ &= -(d-3)u\left[(d-3)(2-3u) - (d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}}u^{\frac{d-1}{d-3}}\right] \\
\mathcal{C}_+ &= \frac{d-2}{2}\left[d-4-(2d-5)u-(d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}}u^{\frac{d-1}{d-3}}\right]
\end{aligned}$$

Interestingly, the zeroth order wavefunction F_0 is an eigenfunction of $\mathcal{H}_+$,

$$\mathcal{H}_+ F_0 = -(d-2)F_0 \tag{6.80}$$

therefore the first-order finite-size effect is a simple shift of the angular momentum operator

$$\hat{L}^2 \to \hat{L}^2 - \frac{d-2}{r_+^2} \tag{6.81}$$

The QNMs of lowest frequency are modified to

$$\omega_0 = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_+} + \mathcal{O}(1/r_+^2) \tag{6.82}$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i\frac{(\ell-1)(\ell+2)}{3r_+} \quad , \quad -i\frac{(\ell+1)^2-4}{4r_+} \tag{6.83}$$

in agreement with numerical results.

One deduces from (6.82) the maximum lifetime of the vector modes,

$$\tau_{\max} = \frac{4\pi}{d}\, T_H \tag{6.84}$$

In the case of a flat horizon ($K = 0$),

$$\omega_0 = -i\frac{k^2}{(d-1)r_+} \tag{6.85}$$

which leads to the diffusion constant

$$D = \frac{1}{4\pi T_H} \tag{6.86}$$

In the case of a hyperbolic horizon ($K = -1$), a similar calculation yields

$$\omega_0 = -i\frac{\xi^2 + \frac{(d-1)^2}{4}}{(d-1)r_+} \quad , \quad \tau = \frac{1}{|\omega_0|} < \frac{16\pi}{(d-1)^2}\, T_H \tag{6.87}$$

It follows that for $d = 5$, these modes live longer than their spherical counterparts which is important for plasma behavior.

6.2.6 Scalar perturbations

Next I consider scalar perturbations which are calculationally more involved but phenomenologically more important because their spectrum contains the lowest frequencies and therefore the longest living modes. For a scalar perturbation we ought to replace the potential $\hat{V}_{\mathsf{V}}$ by

$$\hat{V}_{\mathsf{S}}(u) = \frac{\hat{f}(u)}{4}\left[\hat{m} + \left(1+\frac{1}{r_+^2}\right)u\right]^{-2}$$
$$\times\left\{d(d-2)\left(1+\frac{1}{r_+^2}\right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}\left(1+\frac{1}{r_+^2}\right)u^{\frac{d-5}{d-3}}\right.$$
$$+(d-4)(d-6)\hat{m}^2u^{-\frac{2}{d-3}} + (d-2)^2\left(1+\frac{1}{r_+^2}\right)^3 u^3$$
$$+2(2d^2-11d+18)\hat{m}\left(1+\frac{1}{r_+^2}\right)^2 u^2$$
$$+\frac{(d-4)(d-6)\left(1+\frac{1}{r_+^2}\right)^2}{r_+^2}u^2 - 3(d-2)(d-6)\hat{m}^2\left(1+\frac{1}{r_+^2}\right)u$$
$$\left.-\frac{6(d-2)(d-4)\hat{m}\left(1+\frac{1}{r_+^2}\right)}{r_+^2}u + 2(d-1)(d-2)\hat{m}^3 + d(d-2)\frac{\hat{m}^2}{r_+^2}\right\} \tag{6.88}$$

where $\hat{m} = 2\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_+^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_+^2}$.

In the large black hole limit $r_+ \to \infty$ with $\hat{m}$ fixed (small), the potential simplifies to

$$\hat{V}_{\mathsf{S}}^{(0)}(u) = \frac{1-u^{\frac{d-1}{d-3}}}{4(\hat{m}+u)^2}\left\{d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d-5}{d-3}}\right.$$
$$+(d-4)(d-6)\hat{m}^2u^{-\frac{2}{d-3}} + (d-2)^2u^3$$
$$\left.+2(2d^2-11d+18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2u + 2(d-1)(d-2)\hat{m}^3\right\} \tag{6.89}$$

The wave equation has an additional singularity due to the double pole of the scalar potential at $u = -\hat{m}$. It is desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential,

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} \frac{u^{\frac{d-4}{2(d-3)}}}{\hat{m}+u} F(u) \tag{6.90}$$

Then the wave equation reads

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega}}F = 0 \tag{6.91}$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m}+u}\right] \\
&\quad -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega}} &= -u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \left[-\frac{(d-2)(d-4)}{4u^2} - \frac{(d-3)(d-4)}{u(\hat{m}+u)} + \frac{2(d-3)^2}{(\hat{m}+u)^2}\right] \\
&\quad - \left[\left\{d-4-(2d-5)u^{\frac{d-1}{d-3}}\right\} u^{\frac{d-5}{d-3}} + 2(d-3)\frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\right] \left[\frac{d-4}{2u} - \frac{d-3}{\hat{m}+u}\right] \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1} \frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad + \frac{\hat{V}_{\mathsf{S}}^{(0)}(u) - \hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

I shall define the zeroth-order wave equation as $\mathcal{H}_0 F_0 = 0$, where

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F' \tag{6.92}$$

The acceptable zeroth-order solution is

$$F_0(u) = 1 \tag{6.93}$$

which is plainly regular at all singular points ($u = 0, 1, -\hat{m}$). It corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$).

The Wronskian is

$$\mathcal{W} = \frac{(\hat{m} + u)^2}{u^{\frac{2d-8}{d-3}}(1 - u^{\frac{d-1}{d-3}})} \tag{6.94}$$

and an unacceptable solution is $\check{F}_0 = \int \mathcal{W}$. It can be written in terms of hypergeometric functions. For $d \geq 6$, it has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-6}{2}} \to \infty$ as $r \to \infty$. For $d = 5$, the acceptable wavefunction behaves as $r^{-1/2}$ whereas the unacceptable one behaves as $r^{-1/2} \ln r$. For $d = 4$, the roles of F_0 and $\check{F}_0$ are reversed, however the results still valid because the correct boundary condition at the boundary is a Robin boundary condition. Finally, note that $\check{F}_0$ is also singular (logarithmically) at the horizon ($u = 1$).

Working as in the case of vector modes, one arrives at the first-order constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{A}\mathcal{W}} = 0 \tag{6.95}$$

because $\mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0) F_0' + \mathcal{C}_{\hat{\omega}} F_0 = \mathcal{C}_{\hat{\omega}}$. This leads to the dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0 \tag{6.96}$$

After some algebra, one obtains

$$\mathbf{a}_0 = \frac{d-1}{2}\,\frac{1+(d-2)\hat{m}}{(1+\hat{m})^2} \quad , \quad \mathbf{a}_1 = \frac{d-3}{(1+\hat{m})^2} \quad , \quad \mathbf{a}_2 = \frac{1}{\hat{m}}\left\{1+O(\hat{m})\right\} \tag{6.97}$$

For small $\hat{m}$, the quadratic equation has solutions

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\;\hat{m} \pm \sqrt{\frac{d-1}{2}\;\hat{m}} \tag{6.98}$$

related to each other by $\hat{\omega}_0^{+} = -\hat{\omega}_0^{-*}$, which is a general symmetry of the spectrum.

Finite size effects at first order amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation

$$\mathbf{a}_0 \to \mathbf{a}_0 + \frac{1}{r_+^2}\mathbf{a}_+ \tag{6.99}$$

After some tedious but straightforward algebra, we obtain

$$\mathbf{a}_+ = \frac{1}{\hat{m}}\left\{1+O(\hat{m})\right\} \tag{6.100}$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\;\hat{m} \pm \sqrt{\frac{d-1}{2}\;\hat{m}+1} \tag{6.101}$$

In terms of the quantum number ℓ,

$$\omega_0^{\pm} \approx -i(d-3)\;\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_+} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}} \tag{6.102}$$

in agreement with numerical results.

Notice that the imaginary part is inversely proportional to r_+, as in vector case.

In the scalar case, we also obtained a finite real part independent of r_+.

The maximum lifetime of a gravitational scalar mode is found from (6.102) to be

$$\tau_{\max} = \frac{d-2}{(d-3)d}\, 4\pi T_H \tag{6.103}$$

In the case of a flat horizon ($K = 0$), one obtains

$$\omega = \pm \frac{k}{\sqrt{d-2}} - i \frac{d-3}{(d-1)(d-2)r_+}\, k^2 \tag{6.104}$$

showing that the speed of sound is

$$v = \frac{1}{\sqrt{d-2}} \tag{6.105}$$

as expected for a CFT and the diffusion constant is

$$D = \frac{d-3}{d-2}\, \frac{1}{4\pi T_H} \tag{6.106}$$

For a hyperbolic horizon ($K = -1$), a similar calculation yields

$$\omega = \pm \sqrt{\frac{\xi^2 + (\frac{d-3}{2})^2}{d-2}} - i \frac{(d-3)[\xi^2 + \frac{(d-1)^2}{4}]}{(d-1)(d-2)r_+} \quad , \quad \tau < \frac{4(d-2)}{(d-3)(d-1)^2}\, 4\pi T_H \tag{6.107}$$

In the physically relevant case $d = 5$, evidently the $K = -1$ scalar modes live longer than any other modes, which is important for plasma behavior.

Chapter 7

Future Research Endeavours

In the ever-evolving landscape of theoretical physics, as we gaze towards the future, the horizon is adorned with intriguing possibilities and avenues for exploration. My forthcoming plans encompass a multifaceted approach, featuring the following interrelated objectives:

1. **Large N Expansion in String Theory:** Investigating the implications and applications of the large N expansion in the context of string theory, unraveling its potential to provide insights into the nature of the universe.

2. **Correlation Functions in Finite-Temperature Field Theory:** Calculating correlation functions for finite-temperature field theories to understand their thermal properties and uncover potential phase transitions.

3. **Gauge/Gravity Correspondence:** This remarkable duality provides a bridge between gravity theories in higher-dimensional space-times and conformal field theories in lower dimensions.

4. In future, I would like to study stability of different black holes and perturbations of Quasi Normal Modes.

5. **Quantum Gravity :** The field of quantum gravity is dynamic and continually evolving, offering a rich landscape for future research. As our understanding of the fundamental nature of spacetime and gravity progresses, several promising avenues emerge for further exploration. The following areas represent potential future research endeavors within the realm of quantum gravity: Quantum Gravity Phenomenology, Cosmological Consequences, Black Hole Information Paradox, Quantum Gravity and Particle Physics, Experimental and Observational Tests.

These future research endeavors represent exciting opportunities to push the boundaries of our current knowledge and pave the way for a deeper understanding of the quantum nature of gravity.

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